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Logistics network represents a complex system where different elements that are logistic locations interact with each other. This interaction contains delays caused by time needed for delivery of the material. In this paper, we study local input-to-state stability of such logistics networks. Their behaviour is described by a functional differential equation with a constant time-delay. An appropriate Lyapunov–Razumikhin function and the small gain condition are utilized to establish some conditions for stability analysis of the network under consideration. Our stability conditions for the logistics network are based on the information about the interconnection properties between logistic locations and their production rates. Finally, numerical results are provided to demonstrate an application of the proposed approach.

**Keywords:** logistics networks; modelling; robust planning; stability



## 1. Introduction

The recent developments in information and communication technologies led to drastic changes in logistics networks. These networks became global, highly dynamic and competitive (Bowersox *et al*. 2006). Thus, survival of a modern logistics network requires development of new techniques and methods for its operation and management applications (Daganzo 2005). Mathematical models provide an opportunity to simulate a logistics system to investigate its properties such as performance, stability, robustness and to design appropriate controls to improve these properties (Silva *et al*. 2006, Wu 2008, Dashkovskiy *et al*. 2009, Cigolini and Grando 2009, Deif and ElMaraghy 2010). Such networks are large-scale interconnected systems with a nonlinear behaviour; moreover they are subject to internal and external 'perturbations' that can destabilize the network and lead to a decrease in performance or a break down. Due to the complexity, such systems are hard to analyze and to control. A decentralized control is in many cases the only one possibility to keep it running. Such basic properties as stability and robustness against disturbances must be assured for a reasonable behaviour of the system. Since reactions on changes happening in the system and the disturbances are usually delayed in time, we will use differential equations with time delays in our approach.

On the other hand, delay-differential equations occur in many areas such as engineering systems, robotics, economics or biological systems. For instance, in economic systems, delays appear in a natural way since decisions and effects are separated by some time interval. The delay effects on the stability of systems is a problem of recurring interest since the delay presence may induce complex and undesired behaviours (oscillation, instability and bad performance) for the schemes (Shi 1998, Niculescu and Gu 2004, Wang *et al*. 2005, Nienhaus and Ziegenbein 2006, Xia *et al*. 2007, Esfanjani *et al*. 2009, Ignaciuk and Bartoszewicz 2009, Wang and Gao 2009). In this paper, we study stability of logistics networks modelled as a system of delay-differential equations as well. By dynamics of the network we consider the change at the stock level of logistic location. This change is caused by in-house production and by the flow of material from other locations. As the locations are usually situated far from each other the flow arrives with a time delay.

Roughly speaking, we call a logistics network stable if the stock level of each its location remains bounded for all times. It is also robust against disturbances (seen as external inputs) if this level

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depends only on the magnitude of the disturbance. This is the desired property that we are going to assure in the logistics networks. To this end we will work in the framework of input-to-state stability (ISS) that was introduced first for the systems without delays by Sontag (1989). ISS for time-delay systems was first considered by Teel (1998). In the same paper, Teel introduces ISS of time-delay systems in terms of Lyapunov functions that are called ISS Lyapunov–Razumikhin functions. A Lyapunov–Razumikhin technique was also considered in Liu and Hill (2009) and Hou and Qian (1998). As an alternative to the Lyapunov–Razumikhin technique, the Lyapunov–Krasovskii functional was introduced in Pepe and Jiang (2006). Though ISS analysis of the logistics networks that includes time-delays in their states (stock levels) is an important problem, so far a very little attention has been paid for the investigation of this problem (Dashkovskiy *et al.* 2009). This motivates this study.

In this paper, we consider the local versions of ISS as in real logistics networks all initial conditions and inputs can be taken uniformly bounded. For the systems without delays there are known small gain conditions that guarantee stability of the interconnected system if each subsystem of a network is ISS (Jiang *et al.* 1996, Dashkovskiy *et al.* 2007, Karafyllis and Jiang 2009, Dashkovskiy *et al.* 2010). These conditions are also applied for time-delays systems as well (Karafyllis and Jiang 2009, Tiwari *et al.* 2009). We are going to apply these conditions to study stability of logistics networks. For illustration, we will consider a typical logistics network and derive explicit conditions for its stability. These conditions will be given in terms of restrictions of maximal production rates of the locations in the network. Finally, simulation results are given to illustrate the usefulness of the proposed approach.

The structure of the paper is as follows. In Section 2, the notions of ISS and the ISS Lyapunov–Razumikhin function for time-delayed systems are presented. The small gains needed for stability analysis are also given. In Section 3, we apply the small gain condition for stability analysis in a typical logistics network that is modelled as a time-delay dynamical system. Simulations results that illustrate the influence of network parameters on the stability are provided in Section 4. Finally, conclusions are presented in Section 5.

## 2. Stability of time-delay systems

The following notions used in this paper are fairly standard.

Let $\mathbb{R}_+$ be the set of nonnegative real numbers, $\mathbb{R}^n_+$ be the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$ and $\mathbb{N}_+ := \{0, 1, 2, \ldots\}$. $x^T$ stands for the transposition of a vector $x \in \mathbb{R}^n$. For $x, y \in \mathbb{R}^n$, we use the partial order induced by the positive orthant. It is given by

$$x \geq y \Leftrightarrow x_i \geq y_i, i = 1, \ldots, n,$$
$$x > y \Leftrightarrow x_i > y_i, i = 1, \ldots, n.$$

We write $x \not\geq y \Leftrightarrow \exists i \in \{1, \ldots, n\} : x_i < y_i$.

For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) = \begin{cases} v(t) & \text{if} \quad t \in [s_1, s_2], \\ 0, & \text{otherwise.} \end{cases}$$

Consider a system that is an interconnection of a set of subsystems with state $x := (x_1^T, \ldots, x_n^T)^T$, given by the states $x_i \in \mathbb{R}^{N_i}, i = 1, \ldots, n$ of the subsystems and denote $N = \sum_{i=1}^n N_i$. Dynamics of the $i$th subsystem $\Sigma_i$ is given by the *functional differential equations*:

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), x_{d1}(t), \ldots, x_{dn}(t), u_i(t)),$$
$$x_i(t) = \xi_i(t) \text{ for } t \in [-T, 0) \tag{1}$$

where $u_i : [-T, +\infty] \to \mathbb{R}^m_+$ is a bounded piecewise continuous external input for $i$th subsystem and $x_j, x_{dj}, j \neq i$ are internal inputs from subsystems $j, j \neq i$ to subsystem $i$, where $x_{dj}(t) := x_j(t - T)$ is given with a delay $T > 0$. Here, we consider a case where all delays are the same. Functions $f_i : \mathbb{R}^{2N+M} \to \mathbb{R}^N_i$ and initial data $\xi_i : \mathbb{R} \to \mathbb{R}^{N_i}_+$ are continuous.

We define $|\cdot|$ some norm in $\mathbb{R}^n$, essential supremum norm of a measurable function $u_i$ by $||u_i||_\infty$, $|x_{di}| := \max_{t-T \leq s \leq t} |x_i(s)|$ and $||x_{di}||_{t_0} := \sup_{s \geq t_0} |x_{di}(s)|$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ with $\alpha(0) = 0$ and $\alpha(t) > 0$ for $t > 0$ is called positive definite. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $K$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $K_\infty$ if, in addition, it is unbounded. Note that for $\gamma \in K_\infty$ the inverse function $\gamma^{-1}$ always exists and $\gamma^{-1} \in K_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $KL$ if, for each fixed $t$, the function $\beta(\cdot, t)$ is of class $K$ and, for each fixed $s$, the function $\beta(s, \cdot)$ is non-increasing and tends to zero for $t \to \infty$. Function $id : \mathbb{R} \to \mathbb{R}$ is such that $id(s) = s$ for all $s \in \mathbb{R}$ and for any functions $f$ and $g$ we denote $f \circ g := f(g)$.

The interconnection $\Sigma$ of subsystems (1) is given by

$$\dot{x} = f(x, x_d, u) = \begin{pmatrix} f_1(x_1, \ldots, x_n, x_{d1}, \ldots, x_{dn}, u_1) \\ \vdots \\ f_n(x_1, \ldots, x_n, x_{d1}, \ldots, x_{dn}, u_n) \end{pmatrix}. \tag{2}$$

We study local input-to-state stability of systems (1) and (2).

**Definition 2.1:** The $i$th subsystem in (2) is called *locally input-to-state stable (LISS)* if there exists $\beta \in KL$, $\gamma \in K_\infty \cup \{0\}$, $r > 0$, such that for any initial data $\xi$, $||\xi||_\infty < r$, any measurable, locally essentially bounded input $u$, the solution exists for all $t \geq 0$ and furthermore it satisfies

$$|x(t)| \leq \max\{\beta(||\xi||_\infty, t), \gamma(||u_{[0,t)}||_\infty)\} \quad (3)$$

Function $\gamma$ is called a nonlinear gain.

**Definition 2.2:** The $i$th subsystem is *LISS* if there exist $\beta_i \in KL$, $\gamma_{ij}$, $\gamma_i \in K_\infty \cup \{0\}$, $r_i > 0$, such that for any $\xi_i$, $||\xi_i||_\infty < r_i$, any essentially bounded inputs $x_j$, $x_{dj}$, $u_i$ the solution exists for all $t \geq 0$ and satisfies

$$|x_i(t)| \leq \max\{\beta_i(||\xi_i||_\infty, t), \max_{j, j\neq i}(||x_{dj[0,t]}||_\infty), \gamma_i(||u_{i[o,t)}||_\infty)\} \quad (4)$$

For stability analysis of system (2) we use the following Razumikhin-type theorem that is an extension of the result in Teel (1998) for local input-to-state stability.

**Theorem 2.3:** *If there exist $\alpha_1, \alpha_2 \in K_\infty$, a continuous function $V : [-T, \infty) \times \mathbb{R}^N \to \mathbb{R}_{\geq 0}$, $\gamma_v, \gamma_u \in K$ and $\alpha_3 \in K$, $\rho$, $\rho_u > 0$ such that $||x_{[-T,0)}||_\infty < \rho$, $||u||_\infty < \rho_u$*

(1) $\alpha_1(|x(t)|) \leq V(t) \leq \alpha_2(|x(t)|)$;
(2) $V(t) \geq \max\{\gamma_v(|V_d(t)|), \gamma_u(|u(t)|)\} \Rightarrow D^+V(t) \leq -\alpha_3(|x(t)|)$, for all, *where* $D^+V(t) := \limsup_{h\to 0^+} \frac{V(t+h)-V(t)}{h}$;
(3) $\gamma_v(s) < s$ *for* $s > 0$;
*then the origin is LISS.*

*Function V is called local input-to-state stable (LISS) Lyapunov–Razumikhin function.*

**Remark 1:** This result can be extended for the case of subsystems with many inputs. See the following theorem.

**Theorem 2.4:** *If there exists $\alpha_{i1}, \alpha_{i2} \in K_\infty$, a continuous function $V_i : [-T, \infty) \times \mathbb{R}^{N_i} \to \mathbb{R}_{\geq 0}$, $\gamma_{iv}, \gamma_{iu}, \gamma_{ij} \in K$ and $\alpha_{i3} \in K$, $\rho_i, \rho_{iu} > 0$ such that $||x_{i[-T,0)}||_\infty < \rho_i$, $||u_i||_\infty < \rho_{iu}$*

(1) $\alpha_{i1}(|x_i(t)|) \leq V_i(t) \leq \alpha_{i2}(|x_i(t)|)$;
(2) $V_i(t) \geq \max\{\gamma_{iv}(|V_{id}(t)|), \max_j \gamma_{ij}(|x_{jd}(t)|), \gamma_{iu}(|u_i(t)|)\} \Rightarrow D^+V_i(t) \leq -\alpha_{i3}(|x_i(t)|)$, for all,
(3) $\gamma_{iv}(s) < s$ *for* $s > 0$;

*then the origin is LISS.*

Note that the existence of the LISS Lyapunov–Razumikhin function for all subsystems $\Sigma_i$ does not guarantee that the whole interconnected system $\Sigma$ will be LISS. To guarantee LISS of system $\Sigma$, we recall a framework for stability analysis of interconnected dynamical systems. This framework was introduced in Dashkovskiy *et al.* (2007, 2010). It is based on the information about the gains $\gamma_{ij}$ that describe the interconnection properties of the system.

Consider operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \begin{pmatrix} \max_{j, j\neq 1} \gamma_{1j}(s_j) \\ \vdots \\ \max_{j, j\neq n} \gamma_{nj}(s_j) \end{pmatrix}, \quad s \in \mathbb{R}^n_+.$$

The next theorem provides a small-gain condition for LISS of interconnection (2) that is an extension of the results in Dashkovskiy *et al.* (2010), Tiwari *et al.* (2009) and Karafyllis and Jiang (2009) for time-delay systems and in Dashkovskiy and Naujok (2010), for local ISS.

**Theorem 2.5:** [*Small gain condition*] *Assume each subsystem $\Sigma_i$ has an LISS Lyapunov–Razumikhin function with corresponding gains $\gamma_{ij}$that are collected in a matrix that defines the operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as above. If this operator satisfies*

$$\Gamma(s) \not\geq s, \quad \forall s, s \neq 0 \quad (5)$$

*then the system $\Sigma$ has a LISS Lyapunov–Razumikhin function and thus is LISS.*

Thus, in the context of this theorem the stability conditions imposed on interconnected systems are particularly restricted to the case when all subsystems are LISS.

Let us apply this theorem for stability analysis of a logistics network given in the following section.

## 3. Application to logistics networks

Consider a logistics network consisted of five locations shown in Figure 1. Delayed differential equations that describe the dynamics of each location are given in the following equation:

$$\begin{aligned} \dot{x}_1(t) &= b_1 f_1(x_1(t-T)) + c_{12} f_2(x_2(t-T)) \\ &\quad + c_{13} f_3(x_3(t-T)) + c_{14} f_4(x_4(t-T) \\ &\quad + c_{15} f_5(x_5(t-T)) - f_1(x_1(t)) \\ &=: \tilde{f}_1(x_1, \ldots, x_5, x_{1d}, \ldots, x_{5d}, u_1 = 0) \\ \dot{x}_i(t) &= u_i(t) + b_i f_i(x_i(t-T)) \\ &\quad + c_{i1} f_1(x_1(t-T)) - f_i(x_i(t)) \\ &=: \tilde{f}_i(x_1, \ldots, x_{5d}, u'i), \quad i = 2, \ldots, 5. \end{aligned} \quad (6)$$

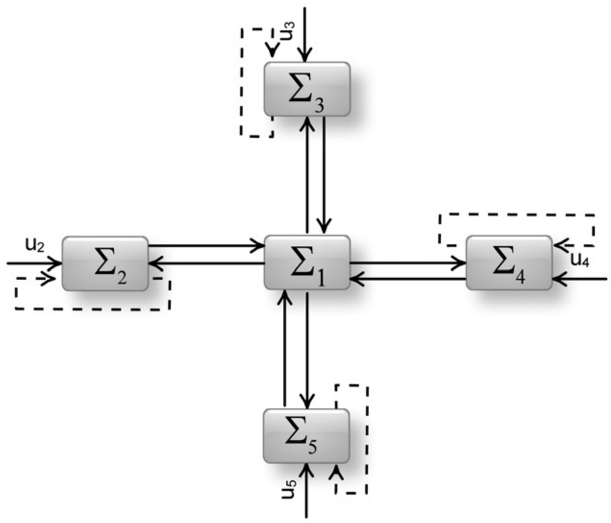


Figure 1. The logistics network.

The logistics network operates as follows. A location $\Sigma_i$ gets material with a delay $T$ from another location, that is, we consider a delay due to transportation that produces a new product with a rate $f_i$ and delivers it then to other locations. The state $x_i$ in (6) is a stock level of a product $i$ of a logistics location $\Sigma_i$, $u_i$ is an external input coming from suppliers and $c_{ji}$ is the share of material produced by location $\Sigma_i$ delivered to location $\Sigma_j$. The component $b_if_i(x_i(t-T))$ determines the returned material to location $i$ after time period $T$ because of rejects, where $b_i$ is the coefficient of rejection. The initial values of stock levels $x_i$ are given by $x_i(t) = \xi_i(t), t \in [-T, 0)$.

A production rate of a location $\Sigma_i$ is given by function $f_i(x_i(t)) := a_i(1 - e^{-x_i(t)})$ where $a_i$ is its maximal production rate. If the stock level $x_i$ is high then the production rate tends to 0. If the stock level is low then the production rate $f_i$ tends to $a_i$. In such a way we intend to keep the stock level low. Thus, we can rewrite system (6) as follows:

$$\begin{aligned}\dot{x}_1(t) &= b_1a_1\big(1-e^{-x_1(t-T)}\big)\\ &\quad + c_{12}a_2\big(1-e^{-x_2(t-T)}\big) + c_{13}a_3\big(1-e^{-x_3(t-T)}\big)\\ &\quad + c_{14}a_4\big(1-e^{-x_4(t-T)}\big)\\ &\quad + c_{15}a_5\big(1-e^{-x_5(t-T)}\big) - a_1\big(1-e^{-x_1(t)}\big)\\ \dot{x}_i(t) &= u_i(t) + b_ia_i\big(1-e^{-x_i(t-T)}\big) + c_{i1}a_1\big(1-e^{-x_1(t-T)}\big)\\ &\quad - a_i\big(1-e^{-x_i(t)}\big), \qquad i = 2,\ldots,5.\end{aligned}$$

It is easy to check that if $\xi_i(t) \geq 0,\ t = [-T, 0)$, then $x_i(t) \geq 0$ for all $t \geq 0, i = 1,\ldots,5$.

Let us recall Definition 2.2 of local input-to-state stability. Then, LISS of location $\Sigma_i$ means, in particular, that for bounded inputs $u_i$ and small initial stock levels $\xi_i$ the stock level for all times is bounded. The boundary depends on the initial stock level $\xi_i$, stock level of the locations $x_{dj[0,t]}$ and external inputs $u_i$. LISS Lyapunov–Razumikhin functions describe how the trajectories of the system behave and allows us to deduce LISS of the system applying Theorem 2.5. In our analysis, we assume that all the locations are LISS and have LISS Lyapunov–Razumikhin functions. However, this is not sufficient for stability of the whole network.

Thus, for stability analysis of the supply chain we need to check the small gain condition (5) using LISS Lyapunov–Razumikhin functions. This condition will impose some restrictions on the cooperation structure of the network.

**Theorem 3.1:** [*Stability condition*] *Consider the logistics network shown in* Figure 1 *and described by* (6). *If*

$$\sum_{k=2}^{5} c_{k1}a_k + b_1a_1 < a_1, \tag{7}$$

*and*

$$||u_j||_\infty + b_ja_j + c_{j1}a_1 < a_j, \quad j = 2,\ldots,5. \tag{8}$$

*hold, then the given logistics network is LISS.*

**Proof:** *Let* $V_i(x_i(t)) = x_i(t), i = 1,\ldots,5,$ *where* $x_i(t) \geq 0$. *The condition* 1) *in Theorem* 4 *is easily satisfied. Let us check the condition* 2). *Taking*

$$\gamma_{1j}(x_j) := -\ln\left(1 - \frac{\sum_{k=2}^{5} c_{1k}a_k + b_1a_1}{(1-\varepsilon_{1j})a_1}(1-e^{-x_j})\right),$$
$$j = 2,\ldots,5, \varepsilon_{1j} \in (0,1) \tag{9}$$

$$\gamma_{1v}(x_1) := -\ln\left(1 - \frac{\sum_{k=2}^{5} c_{1k}a_k + b_1a_1}{(1-\varepsilon_{1v})a_1}(1-e^{-x_1})\right),$$
$$\varepsilon_{1v} \in (0,1). \tag{10}$$

*we obtain that*

$$\begin{aligned}V_1(x_1(t)) = x_1(t) > \ &\max\{\gamma_{12}(x_2(t-T)),\ldots,\\ &\gamma_{15}(x_5(t-T)), \gamma_{1v}(x_1(t-T))\}\\ &\Rightarrow D^+V_1(x_1(t)) \leq -\alpha_{31}(|x_1(t)|),\end{aligned}$$

*where* $\varepsilon_1 := \min_{j=2,\ldots,5,v}\{\varepsilon_{1j}\}$ *and* $\alpha_{31}(|x_1(t)|) := \varepsilon_1a_1\big(1-e^{-x_1(t)}\big)$. *Functions* $\gamma_{1j}, \gamma_{1v} \in K_\infty, j = 2,\ldots,5$ *if*

$$\sum\nolimits_{k=2}^{5} c_{k1}a_k + b_1a_1 < (1-\varepsilon_{1j})a_1 < a_1,$$

*Hence, condition* (2) *of Theorem* 2.4 *holds. Condition* (7) *guaranties also condition* (3); *then the function* $V_1$ *is a local Lyapunov–Razumikhin function for system* $\Sigma_1$.

Taking for $j = 2 \ldots, 5$ the following gain functions

$$\gamma_{j1}(x_1) := -\ln\left(1 - \frac{\|u_j\|_\infty + b_j a_j + c_{j1}a_1}{(1-\varepsilon_{j1})a_j}(1 - e^{-x_1})\right), \quad \varepsilon_{j1} \in (0,1), \tag{11}$$

$$\gamma_{jv}(x_j) := -\ln\left(1 - \frac{\|u_j\|_\infty + b_j a_j + c_{j1}a_1}{(1-\varepsilon_{jv})a_j}(1 - e^{-x_j})\right), \quad \varepsilon_{jv} \in (0,1), \quad j = 2,\ldots,5. \tag{12}$$

we obtain

$$V_2(x_2(t)) = x_2(t) > \max\{\gamma_{21}(x_1(t-T)), \gamma_{2v}(x_2(t-T)), \gamma_{2u}(u(t))\} \Rightarrow D^+V_2(x_2(t)) \le -\alpha_{32}(|x_2(t)|)$$

where $\varepsilon_2 := \min\{\varepsilon_{21}, \varepsilon_{2v}, \varepsilon_{2u}\}$ and $\alpha_{32}(|x_2(t)|) := \varepsilon_2 a_2\left(1 - e^{-x_2(t)}\right)$.

For $\gamma_{j1}, \gamma_{jv} \in K_\infty$ we need

$$\|u_j\|_\infty + b_j a_j + c_{j1}a_1 < a_j, \quad j = 2,\ldots,5.$$

$$\gamma_{ju}(u_j(t)) := -\ln\left(1 - \frac{u_j(t)(\|u_j\|_\infty + b_j a_j + c_{j1}a_1)}{\|u_j\|_\infty(1-\varepsilon_{ju})a_j}\right) \in K, \quad j = 2,\ldots,5. \tag{13}$$

Hence, condition (2) of Theorem 2.4 holds. To satisfy condition (3) we need (8). Then it follows that $V_2$ is a local Lyapunov–Razumikhin function for system $\Sigma_2$. In a similar way, we can check that functions $V_3, \ldots, V_5$ are local Lyapunov–Razumikhin functions for systems $\Sigma_3$, $\Sigma_4$ and $\Sigma_5$ accordingly.

Let us check the small gain condition (5). From Dashkovskiy *et al.* (2007), the inequality $\Gamma \ngeq id$ is equivalent to

$$\gamma_{1j} \circ \gamma_{j1} < id, \quad j = 2,\ldots,5. \tag{14}$$

Consider the left-hand side of the inequality (14)

$$\gamma_{1j} \circ \gamma_{j1}(s) = -\ln\left(1 - \frac{\sum_{k=2}^{5} c_{k1}a_k + b_1 a_1}{(1-\varepsilon_{1j})a_1} \times \frac{\|u_j\|_\infty + b_j a_j + c_{j1}a_1}{(1-\varepsilon_{j1})a_j}(1 - e^{-s})\right) < s.$$

if

$$\frac{\sum_{k=2}^{5} c_{k1}a_k + b_1 a_1}{(1-\varepsilon_{1j})a_1}\frac{\|u_j\|_\infty + b_j a_j + c_{j1}a_1}{(1-\varepsilon_{j1})a_j} < 1. \tag{15}$$

From (7) and (8) there exist such $\varepsilon_{1j}, \varepsilon_{j1} \in (0,1)$ that the inequality (15) holds. Thus, the condition (14) holds and by Theorem 2.5 the whole system is LISS. □

Thus, to design a network that would be robust to perturbations a manager needs to adjust production rates and material shares to satisfy conditions (7) and (8). These conditions require, roughly speaking, that the overall inflow of material to a location was less than its maximal production rate.

In the following section, numerical examples are provided that show influence of the network parameters on stability and application of the stability condition (7) and (8).

## 4. Simulation results

To illustrate the influence of network parameters on the network stability we vary the following parameters of the network in Figure 1: size of the time delay, maximal production rate and share of material flow between locations.

**Example 4.1 (Varying of time delay):** Let us take values $c_{12} = 0.3$, $c_{13} = 0.1$, $c_{14} = 0.3$, $c_{15} = 0.5$, $c_{21} = 0.3$, $c_{31} = 0.1$, $c_{41} = 0.2$, $c_{51} = 0.3$ for the shares of delivered products; $b_1 = 0.05$, $b_2 = 0.06$, $b_3 = 0.05$, $b_4 = 0.1$, $b_5 = 0.03$ for the rate of returned products and $u_2 = 3$, $u_3 = 1$, $u_4 = 2$, $u_5 = 5$, for external inputs.

The initial values of stock level are given by $x_i(t) = 1, i = 1,\ldots,, 5,\ t \in [-T, 0)$. To have the gains well defined, we need to satisfy the condition on maximal production rates, that is, conditions (7) and (8) in Theorem 3.1:

$$c_{12}a_2 + c_{13}a_3 + c_{14}a_4 + c_{15}a_5 + b_1 a_1 < a_1, \quad \|u_j\|_\infty + b_j a_j + c_{j1}a_1 < a_j, \quad j = 2,\ldots,5. \tag{16}$$

Taking maximal production rates $a_1 = 8$, $a_2 = 5.8$, $a_3 = 3$, $a_4 = 5.2$ and $a_5 = 7.8$ we can check these conditions

$$\begin{aligned}
&c_{12}a_2 + c_{13}a_3 + c_{14}a_4 + c_{15}a_5 + b_1 a_1 \\
&\quad = 0.3 \cdot 5.8 + 0.1 \cdot 3 + 0.3 \cdot 5.2 + 0.5 \cdot 7.8 + 0.05 \cdot 8 \\
&\quad = 7.9 < 8, \\
&\|u_2\|_\infty + b_2 a_2 + c_{21}a_1 \\
&\quad = 3 + 0.06 \cdot 5.8 + 0.3 \cdot 8 \\
&\quad = 5.748 < 5.8, \ \rho_{2u} = 3.052, \\
&\|u_3\|_\infty + b_3 a_3 + c_{31}a_1 = 1 + 0.05 \cdot 3 + 0.1 \cdot 8 \\
&\quad = 1.95 < 3, \ \rho_{3u} = 2.05, \\
&\|u_4\|_\infty + b_4 a_4 + c_{41}a_1 = 2 + 0.1 \cdot 5.2 + 0.2 \cdot 8 \\
&\quad = 4.12 < 5.2, \ \rho_{4u} = 3.08, \\
&\|u_5\|_\infty + b_5 a_5 + c_{51}a_1 = 5 + 0.03 \cdot 7.8 + 0.3 \cdot 8 \\
&\quad = 7.634 < 7.8, \ \rho_{5u} = 5.166.
\end{aligned} \tag{17}$$

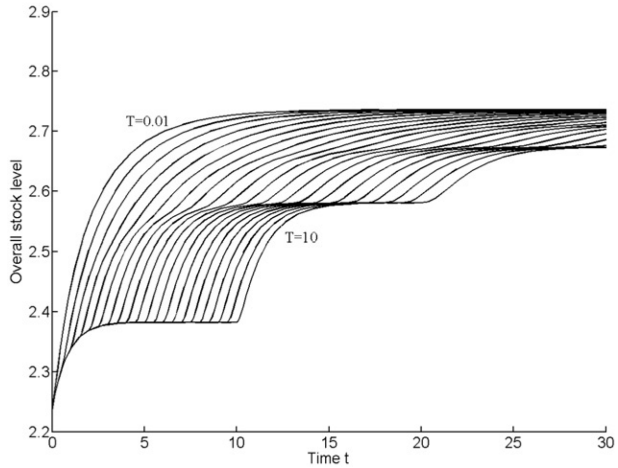


Figure 2. Trajectories of overall stock level for different time delays.

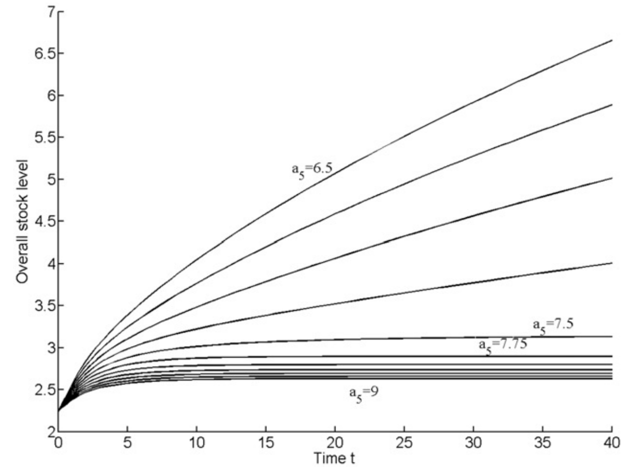


Figure 3. Trajectories of overall stock level for different maximal production rates of location 5.

Thus, the network is LISS. Note that our stability condition does not depend on the size of the time delay $T$. This is also illustrated in Figure 2. The trajectories of the overall stock level for different time delays from $T = 0.01$ to $T = 10$ are shown in this figure and they are bounded. However, for large $T$ the changes in the behaviour of the network occur in a wavy manner due to the late reaction on the changes in stock levels of locations.

**Example 4.2 (Varying of maximal production rate):** Consider the same network as in the previous example and take time delay $T = 0.5$. We investigate the changes in trajectory of the overall stock level under the changes in the maximal production rate of location $\Sigma_5$, that is, in coefficient $a_5$.

In Figure 3 the trajectories are shown for rates $a_5$ from 6.5 to 9. One can mention that for production rates under $a_5 = 7.5$ the networks behaviour is unstable. For such production rates the last condition on production rates in (17) is not satisfied. For example, for $a_5 = 7.5$ we have

$$||u_5||_\infty + b_5 a_5 + c_{51} a_1 = 5 + 0.03 \cdot 7.5 + 0.3 \cdot 8 = 7.625 > 7.5.$$

Thus we cannot apply Theorem 3.1 to establish stability of the network.

**Example 4.3 (Varying of shares of material flow between locations):** In this example, we consider the same parameters as in Example 4.2 and take $a_5 = 7.8$. We investigate the changes in the shares of products to be delivered by location $\Sigma_1$ to locations $\Sigma_2$ and $\Sigma_3$, that is, in coefficients $c_{21}$ and $c_{31}$. In the simulation shown in Figure 4, we vary $c_{21}$ and $c_{31}$ in a such way that their sum stays unchanged and equals 0.45, that is, $c_{21} + c_{31} = 0.45$.

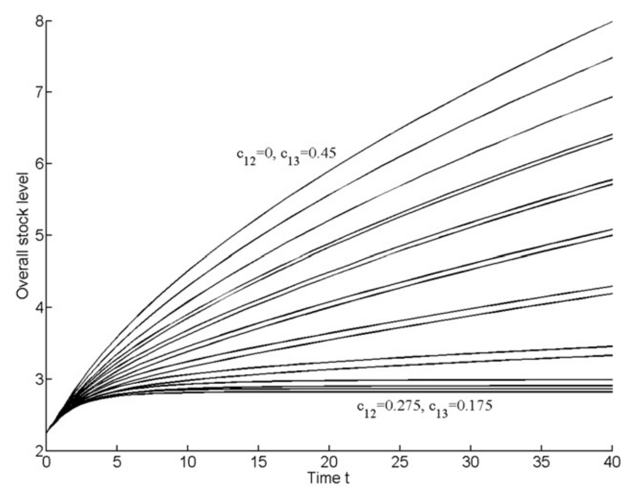


Figure 4. Trajectories of overall stock level for different shares of material flow from location 1 to locations 2 and 3.

The best share in sense of stability is $c_{21} = 0.275, c_{31} = 0.175$ (the lowest trajectory in Figure 4). Furthermore, if we check the second and third conditions in equation (17), then we will see that they are satisfied:

$$||u_3||_\infty + b_3 a_3 + c_{31} a_1 = 1 + 0.06 \cdot 3 + 0.275 \cdot 8 = 5.38 < 5.8,$$

$$||u_3||_\infty + b_3 a_3 + c_{31} a_1 = 1 + 0.05 \cdot 3 + 0.175 \cdot 8 = 2.55 < 3,$$

Thus the network is also LISS by Theorem 3.1.

The most unstable behaviour corresponds to the case $c_{21} = 0, c_{31} = 0.45$, that is, location $\Sigma_1$ delivers all the material to location $\Sigma_3$. For such share of material flow the third condition in equation (17) is not satisfied:

$$||u_3||_\infty + b_3 a_3 + c_{31} a_1 = 1 + 0.05 \cdot 3 + 0.45 \cdot 8 = 4.75 > 3$$

and we cannot apply Theorem 3.1 to establish stability of the network.

## 5. Conclusions

The problem of stability analysis for a class of logistics networks with time delays was investigated in this paper. We have provided a generic approach for modelling of these networks with different production rates of each location and constant time delays in the deliveries. In particular, we model a logistics network as an interconnection of dynamic subsystems with time delays. Subsystems describe behaviour of logistic locations where the state is the stock-level of a location and behaviour is given by a sum of all possible inflows and outflows of a location. An appropriate Lyapunov–Razumikhin function and the small gain condition were utilized to establish some delay-independent conditions on the interconnection and production rates that guarantee local input-to-state stability of the network. This condition requires that all locations possess stable behaviour and the overall inflow of material to a location is less than its maximal production rate. Finally, by way of the numerical examples we have demonstrated application of the stability condition and investigated the influence of different network parameters on stability, namely of time delay, maximal production rate and share of material flow.

Our approach is generic and can be used for larger systems with an arbitrary topology. For example, this approach can be used for design of stable logistics networks such as production networks, delivery networks or global supply chains. In future we plan to investigate delay-dependent stability analysis for the network under consideration by considering an appropriate Lyapunov–Razumikhin function which inserts the delay terms into the function.


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