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Adding an integrator and uniform ISS stabilization for switched MIMO triangular systems with unknown switched signal and right invertible input-output maps

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We extend some recent (2009,2010) results devoted to the Lyapunov stabilization for the switched systems in the strict-feedback form. More specifically, we prove that a multi-input and multi-output triangular switched system with an unknown switching signal, with right-invertible input-output links, and with dynamics, which is affine in external disturbances is globally uniformly input-to-state stabilizable with respect to the disturbances.
С. Дашковський, С.С. Павличков. **Обхід інтегратора і рівномірна стабілізація вхід-стан трикутних систем з переміканнями з багатьма входами і виходами, з невідомими перемиканнями і з правооборотними відображеннями вхід-вихід.** У роботі узагальнюються деякі нові (2009,2010) результати, які присвячені стабілізації за Ляпуновим систем з перемиканнями трикутного вигляду. Точніше, ми доводимо, що трикутні системи з правооборотними відображеннями вхід-вихід і з правими частинами афінними щодо зовнішніх збурень рівномірно стабілізуються за входом-станом відносно збурень.
С. Дашковский, С.С. Павличков. **Обход интегратора и равномерная стабилизация вход-состояние треугольных систем с переключениями со многими входами и выходами, с неизвестными переключениями и с правообратимыми отображениями вход-выход.** В работе обобщаются некоторые новые (2009,2010) результаты, посвященные стабилизации по Ляпунову систем с переключениями треугольного вида. Более точно, мы доказываем, что треугольные системы с неизвестными переключениями, с многими входами и выходами, с правообратимыми отображениями вход-выход и с правыми частями аффинными относительно внешних возмущений равномерно стабилизируемы по входу-состоянию относительно возмущений.
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1. Introduction

In 90th and 2000th various backstepping algorithms (which were originally motivated by earlier (1973) work [5]) were designed and become very fruitful in solving many problems of robust and adaptive nonlinear control. Originally developed for ODE systems [8], this technique was extended to other types of systems (as examples we can mention Volterra systems [6], discrete-time systems [17, 19], or delay systems [2]). On the other hand, during the last decade, switched systems (with or without control) have received a lot of attention. Therefore it is natural to extend well-known recursive designs to the switched systems case.

Some authors made their efforts along this research line during the last years [11, 3, 18, 16, 13]. However, it should be noted that most backstepping designs for switched systems presume that some information about the switching signal is available [3, 18]. Furthermore, sometimes the switching signal is treated as a component of the control input to be designed [11]. The problem of asymptotic stabilization with unknown switched signal was investigated in work [16]. More specifically, in this work it was proven that the classical backstepping design is possible for this class under the so-called "simultaneous domination assumption", which means the existence of a common virtual control and common Lyapunov function at each step of the recursive design. On the other hand, no constructive conditions for verification of the "simultaneous domination assumption" were proposed in [16].

In the most recent work [12] it was proved that the class of strict-feedback form single-input and signle-output (SISO) switched systems considered in [16] indeed does satisfy the "simultaneous domination assumption" and the Lyapunov stabilization is possible (indepenendtly this result was announced in [1]). Thus, in order to stabilize these systems by means of a smooth feedback, one does not need to impose any additional assumptions on this class.

The goal of the current paper is to extend this above-mentioned result to the case of ISS uniform stabilization for the multi-input and multi-output triangular switched system with an unknown switching signal and with right-invertible input-output links.

2. Preliminaries

Throughout the paper, by $\langle\cdot,\cdot\rangle$ we denote the scalar product in $\mathbb{R}^q$ (for any $q\in\mathbb{N}$ and from the context it will be clear which q is considered); for $A\subset\mathbb{R}^q$ by $\overline{A}$ we denote the closure of A. For a vector $\xi\in\mathbb{R}^q$, by $|\xi|$ we denote its quadratic norm, i.e., $|\xi|=\langle\xi,\xi\rangle^{\frac{1}{2}}$.

Also we use the following standard abbreviations: ODE for "ordinary differential equations", MIMO for "multi-input and multi-output", SISO for "single-input and single-output", GAS for "global asymptotic stability/globally asymptotically stable", GES for "global exponential stability/globally exponentially stable", (and respectively LAS and LES for the corresponding local asymptotic or exponential stability), ISS for "input-to-state stability/input-to-state stable".

We say that function α of $[0,+\infty)$ to $[0,+\infty)$ is of class $\mathcal{K}$ iff it is continuous, positive definite and strictly increasing, and α is of class $\mathcal{K}_\infty$ if it is of class $\mathcal{K}$ and unbounded. A function β of $\mathbf{R}_+ \times \mathbf{R}_+$ to $\mathbf{R}_+$ is said to be of class $\mathcal{KL}$ iff for each fixed $t \geq 0$ the function $\beta(\cdot, t)$ is of class $\mathcal{K}_\infty$ and for each fixed $s \geq 0$, we have $\beta(s,t) \to 0$ as $t \to +\infty$ and $t \mapsto \beta(s,t)$ is decreasing.

Consider the nonlinear switched system

$$\dot{x} = F_{\sigma(t)}(t, x, \Delta) \tag{1}$$

where $[0, +\infty) \ni t \mapsto \sigma(t) \in \{1, \ldots, M\}$ is the piecewise constant switching signal, $x \in \mathbb{R}^n$ is the state, $\Delta(t) \in \mathbf{R}^N$ is the input, which is treated as an external disturbance. Suppose that each F_σ is continuous w.r.t (t, x, Δ) and satisfies the local Lipschitz condition w.r.t. (x, Δ).

Given any $\Delta(\cdot)$ in $L_\infty([0, +\infty); \mathbf{R}^N)$ by $\|\Delta(\cdot)\|$ denote its L_∞ - norm on $[0, +\infty[$, and for each $x^0 \in \mathbb{R}^n$ and each $t_0 \geq 0$ and each piecewise constant $\sigma(\cdot)$ by $x(t, x^0, t_0, \Delta(\cdot), \sigma(\cdot))$ denote solution of the Cauchy problem $x(t_0) = x^0$, of system (1) with these $\Delta = \Delta(t)$ and $\sigma = \sigma(t)$.

Definition 1 *System (1) is said to be uniformly input-to-state stable (ISS) iff there are $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that for each t_0, each $x^0 \in \mathbb{R}^n$ and each $\Delta(\cdot) \in L_\infty([t_0, +\infty); \mathbf{R}^N)$ and each piece-wise constant $t \mapsto \sigma(t) \in \{1, \ldots, M\}$ we obtain for all $t \geq t_0$*

$$|x(t, x^0, t_0, \Delta(\cdot), \sigma(\cdot))| \leq \beta(|x^0|, t - t_0) + \gamma(\| \Delta(\cdot)\|_{L_\infty[t_0, +\infty[}) \tag{2}$$

Definition 2 *In the special case, if system (1) has the form*

$$\dot{x} = F_{\sigma(t)}(x, \Delta)$$

it is said to be uniformly input-to-state stable (ISS) iff there are $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that for each $x^0 \in \mathbb{R}^n$, each $\Delta(\cdot) \in L_\infty([0, +\infty); \mathbf{R}^N)$ and each piecewise constant $t \mapsto \sigma(t) \in \{1, \ldots, M\}$ we obtain

$$|x(t, x^0, 0, \Delta(\cdot), \sigma(\cdot))| \leq \beta(|x^0|, t) + \gamma(\| \Delta(\cdot)\|_{L_\infty}), \quad t \geq 0. \tag{3}$$

Remark 1. It is clear that in the special case $\Delta(t) \equiv 0$ the uniform ISS property implies that, whatever piecewise constant $\sigma(\cdot)$ is, $x(t) \equiv 0$ is a GAS solution of the obtained ODE system (1).

Remark 2. Note that this definition is coordinate-free in the following sense: if $z = F(x)$ is a global diffeomorphism of the state space then the definition of the uniform ISS property of system (1) is invatriant w.r.t. all such state transformations. The proof is the same as for the classical ISS concept introduced in [15] and is based on the properties of $\gamma \in \mathcal{K}$ and $\beta \in \mathcal{KL}$ functions.

Definition 3 *Similarly to the ODE case, given a differentiable function $V(t, x)$, define its derivative w.r.t. system (1) as*

$$\frac{d}{dt} V(t, x)|_{(1)} = \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} F_\sigma(t, x, \Delta)$$

for all $\sigma \in \{1, \dots, M\}$ *and all* (t, x, Δ) *(in the special case* $V = V(x)$ *the first term vanishes). If we deal with a control switched system*

$$\dot{x} = F_{\sigma(t)}(t, x, u, \Delta), \tag{4}$$

with controls $u \in \mathbb{R}^m$ *then we define*

$$\frac{d}{dt}V(t,x)|_{(4)} = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F_\sigma(t, x, u, \Delta)$$

for all $\sigma \in \{1, \dots, M\}$ *and all* (t, x, u, Δ)*, and, for a certain controller* $u = u(t, x)$*, we denote*

$$\frac{d}{dt}V(t,x)|_{(4),u=u(t,x)} = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} F_\sigma(t, x, u(t,x), \Delta)$$

for all $\sigma \in \{1, \dots, M\}$ *and all* (t, x, Δ) *(the same holds true for the special cases* $u = u(x)$ *and* $u = u(t)$*).*

3. Main result.

We consider the following switched system

$$\dot{X} = A(X, u) + B_{\sigma(t)}(X) + C_{\sigma(t)}(X)d(t), \qquad t \geq 0, \tag{5}$$

where $\sigma(t) \in \{1, \dots, M\}$ is the piecewise constant *unknown* swithching signal, $X = [x_1, ..., x_\nu]^T$, is the state with $x_i \in \mathbb{R}^{m_i}$, i=1, ..., ν, $u = [u_1, ..., u_{m_{\nu+1}}]^T$ in $\mathbb{R}^{m_{\nu+1}}$ is the control, and $d(t) = [d_1(t), \dots, d_N(t)]^T$ is the external input signal considered as some external disturbances.

We assume that $A(X, u)$, $B_\sigma(X)$ and $C_\sigma(X)$ have the following form

$$A(X,u) = \begin{pmatrix} a_1(x_1, x_2) \\ a_2(x_1, x_2, x_3) \\ \dots \\ a_\nu(x_1, x_2, \dots, x_\nu, u) \end{pmatrix}; \quad B_\sigma(X) = \begin{pmatrix} b_{\sigma,1}(x_1) \\ b_{\sigma,2}(x_1, x_2) \\ \dots \\ b_{\sigma,\nu}(x_1, x_2, \dots, x_\nu) \end{pmatrix}$$

$$\text{and} \quad C_\sigma(X) = \begin{pmatrix} c_{\sigma,1}(x_1) \\ c_{\sigma,2}(x_1, x_2) \\ \dots \\ c_{\sigma,\nu}(x_1, x_2, \dots, x_\nu) \end{pmatrix} \tag{6}$$

for all $\sigma = 1, \dots, M$ and satisfy the following conditions:

(a) functions $A(\cdot,\cdot)$, $B_\sigma(\cdot)$, and $C_\sigma(\cdot)$ are of class $C^{\nu+1}$, and $A(0,0) = 0$, $B_\sigma(0) = 0$, and $C_\sigma(0) = 0$ for all $\sigma = 1, \dots, M$;

(b) for each fixed $i = 1, \dots, \nu$ function $a_i(x_1, x_2, \dots, x_i, \cdot)$ is right invertible in the following sense: there exists a function $[x_1, ..., x_i, \xi] \mapsto \chi_i(x_1, ..., x_i, \xi)$ of class $C^{\nu+1}(\mathbb{R}^{(m_1+\dots m_i)+m_i}; \mathbb{R}^{m_{i+1}})$ such that $\chi_i(0, ..., 0, 0) = 0$ and

$$a_i(x_1, ..., x_i, \chi_i(x_1, ..., x_i, \xi)) = \xi \quad \text{for all } (x_1, ..., x_i, \xi) \text{ in } \mathbb{R}^{(m_1+\dots m_i)+m_i}.$$

Remark 3 Assumption (b) holds true for instance if following [5] we assume that $x_i \in \mathbb{R}^1$ and there is $a > 0$ such that $|\frac{\partial a_i(x_1,...,x_{i+1})}{\partial x_{i+1}}| \geq a > 0$ for all $(x_1, ..., x_{i+1})$ in $\mathbb{R}^{i+1}$, $i = 1, \ldots, \nu$. Of course, in the general MIMO case $m_i \geq 1$, if $m_i \leq m_{i+1}$ for all $i = 1, \ldots, \nu$, then there are many systems which satisfy condition (b) as well (while, if $m_i > m_{i+1}$, then by the well-known Sard theorem, Condition (b) becomes impossible). It is also clear that condition (b) is more restrictive than the case of the generalized triangular forms (GTF) introduced (for ODE systems) in [7, 14]. Currently as open challenging problem is that of extension of the result from [14] (as well as those of the current paper) to the GTF switched systems.

Our main result devoted to the switched systems of the form (5) is as follows.

Theorem 1 *Suppose that system (5) with $A(\cdot,\cdot)$, $B_\sigma(\cdot)$, $C_\sigma(\cdot)$ of form (6) satisfies (a),(b). Then there exists a feedback $u = u(X)$ of class C^1 such that the closed-loop system (5) with this $u = u(X)$ is uniformly ISS with respect to the disturbance $d(t)$.*

The proof of Theorem 1 is reduced to the recursive design given in the next section.

4. Adding an integrator and backstepping design.

Consider a switched control system

$$\dot{z} = f(z, z_{k+1}) + \psi_{\sigma(t)}(z) + \varphi_{\sigma(t)}(z)\Delta(t), \quad t \in \mathbb{R} \tag{7}$$

where $z_{k+1} \in \mathbb{R}^{n_k}$ is the control, $z \in \mathbb{R}^n$, is the state, $\Delta(t) \in L_\infty([0, +\infty); \mathbb{R}^N)$ is some external disturbance and $[0, +\infty) \ni t \mapsto \sigma(t) \in \{1, \ldots, M\}$ is the *unknown* piecewise constant switching signal which takes values from some finite set of indices $\{1, \ldots, M\}$.

In addition, we consider its dynamical extension of the form

$$\begin{cases} \dot{z} = f(z, z_{k+1}) + \psi_{\sigma(t)}(z) + \varphi_{\sigma(t)}(z)\Delta(t) \\ \dot{z}_{k+1} = f_{k+1}(z, z_{k+1}, u) + \psi_{k+1,\sigma(t)}(z, z_{k+1}) + \varphi_{k+1,\sigma(t)}(z, z_{k+1})\Delta(t) \end{cases} \tag{8}$$

with states $[z, z_{k+1}]^T$, controls $u \in \mathbb{R}^m$ (and with the same external disturbance $\Delta(t)$ and *unknown* switching signal $\sigma(t)$).

Next we suppose that systems (7)-(8) satisfy the following conditions

(i) functions f, ψ_σ, φ_σ and f_{k+1}, $\psi_{k+1,\sigma}$, $\varphi_{k+1,\sigma}$ are of class C^p and $f(0,0) = \psi_\sigma(0) = 0$, $\varphi_\sigma(0) = 0$ and $f_{k+1}(0,0,0) = \psi_{k+1,\sigma}(0,0) = 0$, $\varphi_{k+1,\sigma}(0,0) = 0$ for all $\sigma \in \{1, \ldots, M\}$

(ii) $f_{k+1}(z, z_{k+1}, \cdot)$ is right invertible for every $[z, z_{k+1}]^T$ in $\mathbb{R}^n \times \mathbb{R}^{n_k}$ in the following sense: there exists the map $\mathbb{R}^n \times \mathbb{R}^{n_k} \times \mathbb{R}^{n_k} \ni [z, z_{k+1}, \xi] \mapsto u(z, z_{k+1}, \xi) \in \mathbb{R}^m$ of class C^p such that $f_{k+1}(z, z_{k+1}, u(z, z_{k+1}, \xi)) = \xi$ for all $[z, z_{k+1}, \xi]$ in $\mathbb{R}^n \times \mathbb{R}^{n_k} \times \mathbb{R}^{n_k}$ and $u(0, 0, 0) = 0$.

Our main Theorem 1 can be reduced to the following statement on adding an integrator.

Theorem 2 *Assume that Conditions (i),(ii) hold true and, for system (7) with the Lyapunov function* $V(z) := \frac{1}{2}\langle z, z\rangle$ *and for every* $\lambda > 0$ *there exists* $\gamma(\cdot) \in \mathcal{K}_\infty \cup \{0\}$ *such that the following inequality holds*

$$\frac{d}{dt}V(z)|_{(7),z_{k+1}=0} = \langle z, f(z,0) + \psi_\sigma(z) + \varphi_\sigma(z)\Delta\rangle \leq -\lambda V(z) + \gamma(|\Delta|)$$

$$\text{for all} \quad z \in \mathbb{R}^n, \ \ \Delta \in \mathbb{R}^N, \ \ \sigma \in \{1,\dots,M\} \tag{9}$$

Then, for system (8) with the Lyapunov function

$$V_{k+1}(z, z_{k+1}) = \frac{1}{2}\langle z, z\rangle + \frac{1}{2}\langle z_{k+1}, z_{k+1}\rangle,$$

for every $\varepsilon > 0$, *for every* $\delta > 0$ *and for* $\gamma_{k+1}(\cdot) \in \mathcal{K}_\infty$ *given by* $\gamma_{k+1}(|\Delta|) = \gamma(|\Delta|) + \delta|\Delta|^2$, *there exists a feedback* $u(z, z_{k+1})$ *of class* C^{p-1} *such that* $u(0,0) = 0$ *and such that the following Lyapunov inequality holds true*

$$\frac{d}{dt}V_{k+1}(z, z_{k+1})|_{(8),u=u(z,z_{k+1})} \leq -(\lambda - \varepsilon)V_{k+1}(z, z_{k+1}) + \gamma_{k+1}(|\Delta|)$$

$$\text{for all} \quad z \in \mathbb{R}^n, \ \ z_{k+1} \in \mathbb{R}^{n_k}, \ \ \Delta \in \mathbb{R}^N, \ \ \sigma \in \{1,\dots,M\} \tag{10}$$

If $n = 0$ *(and* $k = 0$*) and system (7) is empty, then the extension (8) becomes*

$$\dot{z}_1 = f_1(z_1, u) + \psi_{1,\sigma(t)}(z_1) + \varphi_{1,\sigma(t)}(z_1)\Delta(t).$$

In this case, if Conditions (i),(ii) hold, we say by definition that the conditions of the current theorem are satisfied with $\gamma(|\Delta|) = 0$ *and the current theorem states the existence of the corresponding feedback* $u(z_1)$ *(*$u(0) = 0$*) which satisfies (10) with* $\gamma_{k+1}(|\Delta|) = \gamma_1(|\Delta|) = \delta|\Delta|^2$.

Let us note that (10) implies the uniform ISS of the system (8), if $0 < \varepsilon < \lambda$. Indeed, take any piecewise constant $[0, +\infty) \ni t \mapsto \sigma(t) \in \{1,\dots,M\}$, any disturbance $\Delta(\cdot) \in L_\infty$ and define $V_{k+1}(t) := V_{k+1}(z(t), z_{k+1}(t))$. Define $\lambda_0 := \lambda - \varepsilon > 0$. Then mulipying

$$\frac{d}{dt}V_{k+1}(t) \leq -\lambda_0 V_{k+1}(t) + \gamma_{k+1}(|\Delta(t)|) \quad \text{for all } t \geq 0$$

by $e^{\lambda_0 t}$ we obtain

$$\frac{d}{dt}(V_{k+1}(\tau)e^{\lambda_0\tau}) \leq \gamma_{k+1}(|\Delta(\tau)|e^{\lambda_0\tau} \quad \text{for all } \tau \geq 0$$

and then integrating we have

$$0 \leq V_{k+1}(t) \leq V_{k+1}(0)e^{-\lambda_0 t} + \int_0^t \gamma_{k+1}(|\Delta(\tau)|)e^{-\lambda_0(t-\tau)}d\tau \quad \text{for all } t \geq 0,$$

i.e.,

$$0 \leq |[z(t), z_{k+1}(t)]|^2 \leq |[z(0), z_{k+1}(0)]|^2 e^{-\lambda_0 t} + \frac{1}{\lambda_0}\gamma_{k+1}(\| \Delta(\cdot)\|_{L_\infty}) \quad \text{for all } t \geq 0,$$

which yields Definition 1.

Let us assume for a moment that Theorem 2 is proved and prove that Theorem 2 implies Theorem 1. The proof of Theorem 2 is given in the next section.

Proof of Theorem 1. The proof is by induction over $i = 1, \dots, \nu$.

(The base case $i = 1$) Fix an arbitrary $\lambda > 0$. For $i = 1$, consider the system

$$\dot{x}_1 = a_1(x_1, x_2) + b_{\sigma(t),1}(x_1) + c_{\sigma(t),1}(x_1)d(t), \qquad t \geq 0 \tag{11}$$

with states $x_1 \in \mathbb{R}^{m_1}$, controls $x_2 \in \mathbb{R}^{m_2}$, external disturbances $d \in \mathbb{R}^N$ and an unknown switching signal $\sigma(t) \in \{1, \dots, M\}$. Take the Lyapunov function $V_1(x_1) = \frac{1}{2}\langle x_1, x_1\rangle$, and any $\delta > 0$. Applying Theorem 2, we obtain the existence of a feedback $x_2 = \alpha_1(x_1)$ of class C^ν such that

$$\frac{d}{dt}V_1(x_1)|_{(11), x_2=\alpha_1(x_1)} \leq -\lambda V_1(x_1) + \delta\langle d, d\rangle \tag{12}$$

for all $\sigma \in \{1, \dots, M\}$, $x_1 \in \mathbb{R}^{m_1}$ and all $d \in \mathbb{R}^N$. At this first step, we consider the (identical) state transformation $z = z_1 := x_1 - \alpha_0 = x_1 - \overline{\alpha}_0$ with $\alpha_0 = \overline{\alpha}_0 = 0$, which brings the dynamics of (11) to the form

$$\dot{z} = \hat{f}(z, x_{i+1}) + \hat{\psi}_{\sigma(t)}(z) + \hat{\varphi}_{\sigma(t)}(z)\Delta, \quad \text{i.e.,} \quad \dot{z} = \hat{f}(z, x_2) + \hat{\psi}_{\sigma(t)}(z) + \hat{\varphi}_{\sigma(t)}(z)\Delta \tag{13}$$

with

$$z = z_1 = x_1; \ \ \hat{f}(z, x_2) = a_1(z, x_2); \ \ \hat{\psi}_\sigma(z) = b_{\sigma,1}(z), \ \ \Delta = d, \quad \hat{\varphi}_\sigma = c_{\sigma,1}(z)$$

Then from (12), we obtain

$$\frac{d}{dt}(V_1(z))|_{(13), x_2=\alpha_1(z)} = -\lambda V_1(z) + \delta\langle \Delta, \Delta\rangle \ \text{ with } \ V_1(z) = \frac{1}{2}\langle z, z\rangle. \tag{14}$$

(The inductive step $i \to (i+1)$) Suppose that for any $\lambda > 0$, and an arbitrarily small $\delta > 0$ there exist functions $\alpha_0 = 0$, $\alpha_1(z_1), \dots, \alpha_i(z_1, ..., z_i)$, $\overline{\alpha}_0 = 0$, $\overline{\alpha}_1(x_1), \dots, \overline{\alpha}_i(x_1, ..., x_i)$ of classes $C^{\nu+1}$, C^ν, $\dots$, $C^{\nu-i+1}$, respectively such that the following conditions hold:

1. $\alpha_i(0) = 0$ and $\overline{\alpha}_i(0) = 0$ and the coordinate transformation

$$\begin{cases} z_1 = x_1 - \alpha_0 \\ z_2 = x_2 - \alpha_1(z_1) \\ \dots \\ z_i = x_i - \alpha_i(z_1, ..., z_{i-1}) \end{cases} \iff \begin{cases} z_1 = x_1 - \overline{\alpha}_0 \\ z_2 = x_2 - \overline{\alpha}_1(x_1) \\ \dots \\ z_i = x_i - \overline{\alpha}_i(x_1, ..., x_{i-1}) \end{cases} \tag{15}$$

brings the dynamics of the system

$$\begin{cases} \dot{x}_1 = a_1(x_1, x_2) + b_{\sigma(t),1}(x_1) + c_{\sigma(t),1}(x_1)d(t) \\ \dot{x}_2 = a_2(x_1, x_2, x_3) + b_{\sigma(t),2}(x_1, x_2) + c_{\sigma(t),2}(x_1, x_2)d(t) \\ \dots \\ \dot{x}_i = a_i(x_1, x_2, ..., x_{i+1}) + b_{\sigma(t),i}(x_1, x_2, ..., x_i) + c_{\sigma(t),i}(x_1, x_2, ..., x_i)d(t) \end{cases} \tag{16}$$

with states $[x_1, ..., x_i]^T$, controls x_{i+1} and external disturbances d to the form

$$\dot{z} = \hat{f}(z, x_{i+1}) + \hat{\psi}_{\sigma(t)}(z) + \hat{\varphi}_{\sigma(t)}(z)\Delta(t) \tag{17}$$

where $z = [z_1, ..., z_i]$ is the state defined by the transformation (15), x_{i+1} is the control, $\Delta(\cdot) = d(\cdot)$ is the disturbance, $\sigma(t) \in \{1, \dots, M\}$ is the unknown (piecewise constant) switching signal.

2. For the Lyapunov function $V_i(z) = \frac{1}{2}\langle z, z\rangle$ we obtain

$$\frac{d}{dt}(V_i(z))|_{(17), x_{i+1}=\alpha_i(z_1,\dots,z_i)} = -\lambda V_i(z) + \delta\langle d, d\rangle \tag{18}$$

for all z, d and all $\sigma \in \{1, ..., M\}$.

Let the system

$$\begin{cases} \dot{z} = \hat{f}(z, x_{i+1}) + \hat{\psi}_{\sigma(t)}(z) + \hat{\varphi}_{\sigma(t)}(z)\Delta(t) \\ \dot{x}_{i+1} = \hat{f}_{i+1}(z, x_{i+1}, w) + \hat{\psi}_{i+1,\sigma(t)}(z, x_{i+1}) + \hat{\varphi}_{i+1,\sigma(t)}(z, x_{i+1})\Delta(t) \end{cases} \tag{19}$$

(with states $[z, x_{i+1}]$, controls $w \in \mathbb{R}^{m_{i+2}}$ and disturbances $\Delta = d$) be the corresponding dynamical extension obtained by the state transformation

$$\begin{cases} z_1 = x_1 - \alpha_0 \\ z_2 = x_2 - \alpha_1(z_1) \\ \dots \\ z_i = x_i - \alpha_i(z_1, ..., z_{i-1}) \\ x_{i+1} = x_{i+1} \end{cases} \quad \text{i.e.,} \quad \begin{cases} z_1 = x_1 - \overline{\alpha}_0 \\ z_2 = x_2 - \overline{\alpha}_1(x_1) \\ \dots \\ z_i = x_i - \overline{\alpha}_i(x_1, ..., x_{i-1}) \\ x_{i+1} = x_{i+1} \end{cases}$$

of the system

$$\begin{cases} \dot{x}_1 = a_1(x_1, x_2) + b_{\sigma(t),1}(x_1) + c_{\sigma(t),1}(x_1)d(t) \\ \dot{x}_2 = a_2(x_1, x_2, x_3) + b_{\sigma(t),2}(x_1, x_2) + c_{\sigma(t),2}(x_1, x_2)d(t) \\ \dots \\ \dot{x}_i = a_i(x_1, x_2, ..., x_{i+1}) + b_{\sigma(t),i}(x_1, x_2, ..., x_i) + c_{\sigma(t),i}(x_1, x_2, ..., x_i)d(t) \\ \dot{x}_{i+1} = a_{i+1}(x_1, x_2, ..., x_{i+1}, w) + b_{\sigma(t),i+1}(x_1, x_2, ..., x_{i+1}) + \\ +c_{\sigma(t),i+1}(x_1, x_2, ..., , x_{i+1})d(t) \end{cases}$$

(with states $[x_1, ..., x_{i+1}]$, controls $w \in \mathbb{R}^{m_{i+2}}$ and disturbances d). Then the state transformation $z = z$, $z_{i+1} = x_{i+1} - \alpha_{i+1}(z)$ brings the system (19) to the form

$$\begin{cases} \dot{z} = f(z, z_{i+1}) + \psi_{\sigma(t)}(z) + \varphi_{\sigma(t)}(z)\Delta(t) \\ \dot{z}_{i+1} = f_{i+1}(z, z_{i+1}, w) + \psi_{\sigma(t),i+1}(z, z_{i+1}) + \varphi_{\sigma(t),i+1}(z, z_{i+1})\Delta(t) \end{cases} \tag{20}$$

(with states $[z, z_{i+1}]$, controls $w \in \mathbb{R}^{m_{i+2}}$ and disturbances $\Delta(t) = d(t)$) By the induction hypothesis (more specifically by the definition of α_i and by (18)), system (20) satisfies all the assumptions of Theorem 2. Applying Theorem 2, for each $\lambda > 0$, each $\varepsilon > 0$ and each $\delta > 0$, we obtain the existence of a feedback $w = \alpha_{i+1}(z, z_{i+1})$ of class $C^{\nu-i}$ with $\alpha_{i+1}(0,0) = 0$ such that the Lyapunov function $V_{i+1}(z) = V_i(z) + \frac{1}{2}\langle z_{i+1}, z_{i+1}\rangle$ satisfies the inequality

$$\frac{d}{dt}(V_{i+1}(z, z_{i+1}))|_{(20), w=\alpha_{i+1}(z_1,\dots,z_i,z_{i+1})} = -(\lambda - \varepsilon)V_{i+1}(z, z_{i+1}) + \delta\langle d, d\rangle \quad (21)$$

Thus, using the induction over $i = 1, \dots, \nu$, we obtain for $i = \nu$ and for every $\lambda > 0$ and $\delta > 0$ that there exist $2(\nu + 1)$ functions $\alpha_0 = 0$, $\alpha_1(z_1)$, ..., $\alpha_\nu(z_1, ..., z_\nu)$, and $\overline{\alpha}_0 = 0$, $\overline{\alpha}_1(x_1)$, ..., $\overline{\alpha}_\nu(x_1, ..., x_\nu)$, such that α_i and $\overline{\alpha}_i$ are of class $C^{\nu-i+1}$ and $\alpha_i(0) = \overline{\alpha}_i(0) = 0$ and such that the following two properties hold

1. The state transformation

$$\begin{cases} z_1 = x_1 - \alpha_0 \\ z_2 = x_2 - \alpha_1(z_1) \\ \dots \\ z_\nu = x_\nu - \alpha_{\nu-1}(z_1, ..., z_{\nu-1}) \end{cases} \iff \begin{cases} z_1 = x_1 - \overline{\alpha}_0 \\ z_2 = x_2 - \overline{\alpha}_1(x_1) \\ \dots \\ z_\nu = x_\nu - \overline{\alpha}_{\nu-1}(x_1, ..., x_{\nu-1}) \end{cases} \quad (22)$$

or in vector form

$$Z = X - \alpha(Z), \iff Z = X - \overline{\alpha}(X) \quad (23)$$

brings the dynamics of the system (5) to the form

$$\dot{Z} = F(Z, u) + Q_{\sigma(t)}(Z) + R_{\sigma(t)}(Z)d(t), \quad (24)$$

with states $Z \in \mathbb{R}^{m_1+\dots+m_\nu}$, controls $u \in \mathbb{R}^{m_{\nu+1}}$ and external disturbances $d(t)$, where $F(Z, u)$, $Q_\sigma(Z)$, and $R_\sigma(Z)$ are defined by the transformation (22) as follows

$$F(Z, u) = A(Z + \alpha(Z), u) - \frac{\partial\overline{\alpha}(Z + \alpha(Z))}{\partial X} A(Z + \alpha(Z), u), \quad (25)$$

$$Q_\sigma(Z) = B_\sigma(Z + \alpha(Z)) - \frac{\partial\overline{\alpha}(Z + \alpha(Z))}{\partial X} B_\sigma(Z + \alpha(Z)), \quad (26)$$

$$R_\sigma(Z) = C_\sigma(Z + \alpha(Z)) - \frac{\partial\overline{\alpha}(Z + \alpha(Z))}{\partial X} C_\sigma(Z + \alpha(Z)). \quad (27)$$

2. For the Lyapunov function $V_\nu(Z) := \frac{1}{2}\langle Z, Z\rangle$ with Z=$[z_1, ..., z_\nu]$ given by (22) we have

$$\frac{d}{dt}(V_\nu(Z))|_{(24), u=\alpha_\nu(Z)} = -\lambda V_\nu(Z) + \delta\langle d, d\rangle \quad (28)$$

Thus, the control u=$\alpha_\nu(Z)$ uniformly ISS stabilizes (24). Taking the state transformation $Z \mapsto X$ which is inverse to (22), we obtain the statement of our Theorem 1.

5. Proof of Theorem 2

Next we use the following straightforward equality

$$\langle z, f(z, z_{k+1}) - f(z, 0)\rangle = \int_0^1 \frac{d}{d\theta}\langle z, f(z, \theta z_{k+1})\rangle d\theta = \langle z, J(z, z_{k+1})z_{k+1}\rangle$$

where

$$J(z, z_{k+1}) = \int_0^1 \frac{\partial f}{\partial x_{k+1}} f(z, \theta z_{k+1}) d\theta \tag{29}$$

Therefore

$$\frac{d}{dt}V_{k+1}(z, z_{k+1})|_{(8),u} = \langle z, f(z, z_{k+1}) + \psi_\sigma(z) + \varphi_\sigma(z)\Delta\rangle +$$
$$+\langle z_{k+1}, f_{k+1}(z, z_{k+1}, u) + \psi_{k+1,\sigma}(z, z_{k+1}) + \varphi_{k+1,\sigma}(z, z_{k+1})\Delta\rangle =$$
$$= \langle z, f(z, 0) + \psi_\sigma(z) + \varphi_\sigma(z)\Delta\rangle + \langle z, J(z, z_{k+1})z_{k+1}\rangle$$
$$+\langle z_{k+1}, f_{k+1}(z, z_{k+1}, u) + \psi_{k+1,\sigma}(z, z_{k+1}) + \varphi_{k+1,\sigma}(z, z_{k+1})\Delta\rangle \leq$$
$$\leq -\lambda V(z) + \gamma(|\Delta|) + \langle z_{k+1}, f_{k+1}(z, z_{k+1}, u) + J^T(z, z_{k+1})z + \psi_{k+1,\sigma}(z, z_{k+1}) +$$
$$+\varphi_{k+1,\sigma}(z, z_{k+1})\Delta\rangle. \tag{30}$$

Our next goal is to find a controller $u = u(z, z_{k+1})$ such that the last term in (30) will provide (10) with the new gain γ_{k+1} mentioned in the statement of Theorem 2 to be proved.

By definition we denote the components of z and z_{k+1} as follows:

$$z = [z^1, \dots, z^n]^T \quad \text{and} \quad z_{k+1} = [z^1_{k+1}, \dots, z^{n_{k+1}}_{k+1}]^T.$$

Using the well-known Hadamard lemma we obtain the existence of matrix functions $\Psi_{k+1,\sigma}(z, z_{k+1})$ and $\Phi_{k+1,\sigma}(z, z_{k+1})$ of class C^{p-1} such that

$$\psi_{k+1,\sigma}(z, z_{k+1}) = \Psi_{k+1,\sigma}(z, z_{k+1}) \begin{bmatrix} z \\ z_{k+1} \end{bmatrix} \quad \text{and}$$

$$\varphi_{k+1,\sigma}(z, z_{k+1}) = \Phi_{k+1,\sigma}(z, z_{k+1}) \begin{bmatrix} z \\ z_{k+1} \end{bmatrix} \quad \text{for all } \sigma = 1, \dots, M. \tag{31}$$

Next we estimate (30) by using (31). To make the estimates shorter, we omit the arguments (z, z_{k+1}) of $\Psi_{k+1,\sigma}(z, z_{k+1})$ and $\Phi_{k+1,\sigma}(z, z_{k+1})$.

Take an arbitrary $\varepsilon > 0$. From the first equality of (31) it follows that for every $z \in \mathbb{R}^n$, every $z_{k+1} \in \mathbb{R}^{n_k}$ and every $\sigma \in \{1, \dots, M\}$ we obtain

$$\left\langle z_{k+1}, \Psi_{k+1,\sigma}(z, z_{k+1}) \begin{bmatrix} z \\ z_{k+1} \end{bmatrix} \right\rangle =$$

$$= \sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n}} \frac{1}{\varepsilon} z^i_{k+1}\Psi^{i,j}_{k+1,\sigma}\varepsilon z^j + \sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n_{k+1}}} z^i_{k+1}\overline{\Psi}^{i,j}_{k+1,\sigma} z^j_{k+1} \le$$

$$\le \frac{1}{2} \sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n}} \left(\frac{1}{\varepsilon^2}\left(z^i_{k+1}\right)^2\left(\Psi^{i,j}_{k+1,\sigma}\right)^2 + \varepsilon^2 (z^j)^2\right) +$$

$$+\frac{1}{2} \sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n_{k+1}}} \left(\left(z^i_{k+1}\right)^2 + \left(\overline{\Psi}^{i,j}_{k+1,\sigma}\right)^2 (z^j_{k+1})^2\right) \le$$

$$\le \frac{1}{2}\sum_{\sigma=1}^{M} \sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n}} \frac{1}{\varepsilon^2} z^i_{k+1} z^i_{k+1}\left(\Psi^{i,j}_{k+1,\sigma}\right)^2 + \frac{1}{2}\sum_{\substack{1\le i\le n_{k+1}\\ 1\le j\le n}} \varepsilon^2 (z^j)^2 +$$

$$+\frac{1}{2} n_{k+1} \sum_{1\le i\le n_{k+1}} \left(z^i_{k+1}\right)^2 + \frac{1}{2}\sum_{\sigma=1}^{M}\sum_{j=1}^{n_{k+1}} z^j_{k+1} \sum_{i=1}^{n_{k+1}} \left(\overline{\Psi}^{i,j}_{k+1,\sigma}\right)^2 z^j_{k+1} =$$

$$= \frac{1}{2}\varepsilon^2 n_{k+1}\langle z, z\rangle + \langle z_{k+1}, H(z, z_{k+1})\rangle, \quad \text{where } H^i(z, z_{k+1}) =$$

$$= \frac{1}{2}\sum_{\sigma=1}^{M}\sum_{j=1}^{n} \frac{1}{\varepsilon^2} z^i_{k+1}\left(\Psi^{i,j}_{k+1,\sigma}\right)^2 + \frac{1}{2} n_{k+1} z^i_{k+1} + \frac{1}{2}\sum_{\sigma=1}^{M}\sum_{j=1}^{n_{k+1}} \left(\overline{\Psi}^{j,i}_{k+1,\sigma}\right)^2 z^i_{k+1}, \quad (32)$$

where $\Psi^{i,j}_{k+1,\sigma}$ and $\overline{\Psi}^{i,j}_{k+1,\sigma}$ denote the corresponding components of $\Psi_{k+1,\sigma}$.

Similarly, from the second equality of (31), for every $\delta > 0$ and for each $z \in \mathbb{R}^n$, each $z_{k+1} \in \mathbb{R}^{n_k}$ and each $\sigma \in \{1, \ldots, M\}$ we obtain

$$\left\langle z_{k+1}, \Phi_{k+1,\sigma}(z, z_{k+1}) \begin{bmatrix} z \\ z_{k+1} \end{bmatrix} \Delta \right\rangle =$$

$$= \sum_{\substack{1\le i\le n_{k+1}\\ 1\le l\le N\\ 1\le j\le n}} \frac{1}{\delta} z^i_{k+1}\delta\Delta_l \Phi^{i,j}_{k+1,\sigma,l} z^j + \sum_{\substack{1\le i\le n_{k+1}\\ 1\le l\le N\\ 1\le j\le n_{k+1}}} \frac{1}{\delta} z^i_{k+1}\delta\Delta_l \overline{\Phi}^{i,j}_{k+1,\sigma,l} z^j_{k+1} \le$$

$$\le \frac{1}{2} \sum_{\substack{1\le i\le n_{k+1}\\ 1\le l\le N\\ 1\le j\le n}} \left(\delta^2\Delta_l^2 + \; + z^i_{k+1}\frac{1}{\delta^2} z^i_{k+1}\left(\Phi^{i,j}_{k+1,\sigma,l} z^j\right)^2\right) +$$

$$+\frac{1}{2}\sum_{\substack{1\le i\le n_{k+1}\\ 1\le l\le N\\ 1\le j\le n_{k+1}}}\left(\delta^2\Delta_l^2+z_{k+1}^i\frac{1}{\delta^2}z_{k+1}^i\left(\overline{\Phi}_{k+1,\sigma,l}^{i,j}z_{k+1}^j\right)^2\right)\le$$

$$\le n_{k+1}(n+n_{k+1})\delta^2\sum_{l=1}^{N}\Delta_l^2+\frac{1}{2}\sum_{i=1}^{n_{k+1}}z_{k+1}^i\sum_{\substack{1\le l\le N\\ 1\le j\le n\\ 1\le\sigma\le M}}\frac{1}{\delta^2}z_{k+1}^i\left(\Phi_{k+1,\sigma,l}^{i,j}z^j\right)^2+$$

$$+\frac{1}{2}\sum_{i=1}^{n_{k+1}}z_{k+1}^i\sum_{\substack{1\le l\le N\\ 1\le j\le n_{k+1}\\ 1\le\sigma\le M}}\frac{1}{\delta^2}z_{k+1}^i\left(\overline{\Phi}_{k+1,\sigma,l}^{i,j}z_{k+1}^j\right)^2, \tag{33}$$

where $\Phi_{k+1,\sigma,l}^{i,j}$ and $\overline{\Phi}_{k+1,\sigma,l}^{i,j}$ denote the corresponding components of $\Phi_{k+1,\sigma}$.

Summing up (30)-(33), we obtain

$$\frac{d}{dt}V_{k+1}(z,z_{k+1})|_{(8),u}\le-(\lambda-\frac{1}{2}\varepsilon^2n_{k+1})V(z)+\gamma(|\Delta|)+n_{k+1}(n+n_{k+1})\delta^2|\Delta|^2+$$

$$+\langle z_{k+1},f_{k+1}(z,z_{k+1},u)+J^T(z,z_{k+1})z+H(z,z_{k+1})+G(z,z_{k+1})\rangle,\quad\text{where}$$

$$G^i(z,z_{k+1}):=\frac{1}{2}\sum_{\substack{1\le l\le N\\ 1\le j\le n\\ 1\le\sigma\le M}}\frac{1}{\delta^2}z_{k+1}^i\left(\Phi_{k+1,\sigma,l}^{i,j}z^j\right)^2+$$

$$+\frac{1}{2}\sum_{\substack{1\le l\le N\\ 1\le j\le n_{k+1}\\ 1\le\sigma\le M}}\frac{1}{\delta^2}z_{k+1}^i\left(\overline{\Phi}_{k+1,\sigma,l}^{i,j}z_{k+1}^j\right)^2.$$

By Assumption (ii), there is $u(z,z_{k+1})$ of class C^{p-1} such that $u(0,0)=0$ and

$$f_{k+1}(z,z_{k+1},u(z,z_{k+1}))+J^T(z,z_{k+1})z+H(z,z_{k+1})+G(z,z_{k+1})=$$

$$=-(\lambda-\frac{1}{2}\varepsilon^2n_{k+1})z_{k+1}\quad\text{for all}\quad[z,z_{k+1}]\in\mathbb{R}^n\times\mathbb{R}^{n_k}.$$

(Note that J, H, G are of class C^{p-1} in general (due to the proof of the Hadamrd lemma); therefore the feedback $u(z,z_{k+1})$ obtained is of class C^{p-1} as a composition of functions of class C^p (see Assumption (ii)) and those of class C^{p-1}).

Since $\varepsilon>0$ and $\delta>0$ are chosen arbitrarily small, this completes the proof of Theorem 2.

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