Backstepping for nonsmooth MIMO nonlinear Volterra systems with noninvertible input-output maps ⋆

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Abstract: We prove the global controllability for a class of nonlinear MIMO Volterra systems of the triangular form as well as for their bounded perturbations. In contrast to the related preceeding work [12], we replace the condition of C^1 smoothness, which was essentially used before, with that of local Lipschitzness. Furthermore, we remove the assumption of the invertibility of the input-output interconnections, which was also essential in these preceeding results. In order to solve the problem, we revise the backstepping procedure proposed in these works, and combine it with another method of constructing discontinuous feedbacks proposed for the so-called "generalized triangular form" in the case of ODE [17, 13, 19].

1. INTRODUCTION

During the last two decades such recursive procedures as backstepping-like designs became very popular when solving various problems of adaptive and robust nonlinear control - [5, 6, 16, 9, 14, 15, 21]. It worth mentioning that, despite of the fruitfulness of the backstepping-like algorithms, the most works devoted to them address the triangular or pure-feedback form systems [10]

$$\begin{cases} \dot{x}_i = f_i(x_1, ..., x_{i+1}), \quad i = 1, \ldots, n-1; \\ \dot{x}_n = f_n(x_1, ..., x_n, u) \end{cases} \tag{1}$$

that are feedback linearizable, i.e. to those which satisfy the condition $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$, $i{=}1, ..., n$; or even have the strict-feedback form $\dot{x}_i = b_i x_{i+1} + \theta_i \varphi_i(x_1, ..., x_i)$, $i = 1, \ldots, n-1$; $\dot{x}_n = b_n u + \theta_n \varphi_n(x_1, ..., x_n)$ (with $b_i \neq 0$). Indeed, whatever the problem is (Lyapunov stabilization, adaptive stabilization etc.), the classical version of the backstepping requires system (1) to satisfy the following two properties:

(A) The virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i-th step ($i = 1, \ldots, n$) should be well-defined as an implicit function obtained from some nonlinear equation of the form $f_i(x_1, ..., x_{i+1}) = F_i(t, x_1, ..., x_i)$ to be resolved w.r.t. x_{i+1} where $F_i(t, x_1, ..., x_i)$ is some function of the previous coordinates $x_1, ..., x_i$ (and maybe of t).

(B) Each virtual control $x_{i+1} = \alpha_i(t, x_1, ..., x_i)$ obtained at the i - th step should be smooth enough because one needs to take its derivatives at the next steps $i = 1, ..., n$.

This necessarily leads to the conditions like $|\frac{\partial f_i}{\partial x_{i+1}}| {\neq} 0$, $i{=}1, ..., n$, (to comply with (A)) and like $f_i {\in} C^n$ or $f_i {\in} C^{n-i+1}$ (to comply with (B)).

Works [3, 4, 23, 24, 15, 20] were devoted to the issue of how to obviate the first restriction $|\frac{\partial f_i}{\partial x_{i+1}}| {\neq} 0$, at least for some special cases: when $f_i(x_1, ..., x_{i+1})$ are polynomials w.r.t. x_{i+1} of odd degree (see work [20]); when $f_i = x_{i+1}^p + \varphi_i(x_1, ..., x_i)$ (see works [24, 15] devoted to the problem of global stabilization of such systems into the origin as well as futher works by some of these authors devoted to various adaptive and robust control probelms for this class); partial-state stabilization under the assumption that the "controllable part" satisfies some additional "growth conditions" (see work [23] and conditions (A3),(i),(ii),(iii)); the problem of feedback triangulation under the assumption that the set of regular points is open and dense in the state space - see work [3].

A natural generalization of these cases is the so-called "generalized triangular form" (GTF), when the only assumption is that $f_i(t, x_1, ..., x_i, \cdot)$ is a surjection whereas x_i and u are vectors not necessarily of the same dimension (and the dynamics is of class C^1 or C^n depending of the problem to be explored). In works [13, 19] it was proved that, first, the systems of this class are globally robustly controllable, in particular, their bounded perturbations are globally controllable as well - [13] and, second, they are globally asymptotically stabilizable into every regular point [19]. Note that, although the methods proposed in [11, 12, 13, 19] are called "backstepping", their only common feature with the classical backstepping designs is the induction over the dimension of the system and treatment x_{i+1} as the virtual control at the i-th step; as to the construction, the approach proposed in [11, 12, 13, 19] is absolutely different. This is especially applied to [13]

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and to the preceeding related works [11, 12] devoted to the problem of global robust controllability.

It worth mentioning that, despite of the importance of the Volterra equations in applications, the controllability problem for the Volterra systems was investigated in few works. Works [1, 2] were devoted to the complete controllability of perturbations of linear Volterra systems. In these papers, some natural analogs of the integral criterion of the controllability for linear ODE systems were obtained.

In works [11, 12] the problem of global robust controllability was successively solved for the nonlinear Volterra systems of the triangular form

$$\dot{x}_i = f_i(t, x_1, ..., x_{i+1}) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds,$$
$$i = 1, \ldots, n,$$

(where $x_{n+1} = u$ is the control, and $(x_1, ..., x_n)$ is the state) including the global controllability of their bounded perturbations. Although, as we highlighted above, the inductive construction proposed in these works differs totally from the classical backstepping designs, the following two assumptions, which are similar to (A) and (B), are essential in this construction:

(A') For every $x_1(\cdot), ..., x_i(\cdot)$ of class C^1 the integral equation

$$\dot{x}_i = f_i(t, x_1(t), ..., x_{i+1}(t)) + \int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_{i+1}(s))ds,$$

should be resolvable w.r.t. $x_{i+1}(\cdot)$ on the whole time interval $[t_0, T]$.

(B') The properties of the linearized control systems (and those of the Frechet derivative of the input-output map) were essential, which is why f_i and g_i should be of class C^1 at least.

The goal of the current paper is to remove these restrictions (A') and (B') and to show how a modification of the methods proposed in [13, 19] can be applied to the problem of global controllability of the Volterra systems.

2. PRELIMINARIES

The results of the current paper are concerned with the control systems of the Volterra integro-differential equations

$$\dot{x}(t) = f(t, x(t), u(t)) + \int_{t_0}^{t} g(t, s, x(s))ds, \; t \in I = [t_0, T] \quad (2)$$

where $u \in \mathbf{R}^m = \mathbf{R}^{m_{\nu+1}}$ is the control, $x = (x_1, ..., x_\nu)^T \in \mathbf{R}^n$ is the state with $x_i \in \mathbf{R}^{m_i}$, $m_i \leq m_{i+1}$ and $n = m_1 + ... + m_\nu$, functions f and g have the form

$$f(t, x, u) = \begin{pmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \ldots \\ f_\nu(t, x_1, ..., x_\nu, u) \end{pmatrix} \quad \text{and}$$

$$g(t, s, x) = \begin{pmatrix} g_1(t, s, x_1) \\ g_2(t, s, x_1, x_2) \\ \ldots \\ g_\nu(t, s, x_1, ..., x_\nu) \end{pmatrix} \quad (3)$$

with $f_i \in \mathbf{R}^{m_i}$, $g_i \in \mathbf{R}^{m_i}$ and satisfy the conditions:

(i) $f \in C(I \times \mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$, $g \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$,

(ii) *f and g satisfy the* local *Lipschitz condition w.r.t. (x, u), i.e., for every compact set $K \subset \mathbf{R}^n \times \mathbf{R}^m$ there is $L_K > 0$ such that, for every $(x^1, u^1) \in K$ and every $(x_2, u_2) \in K$ we obtain*

$$|f(t, x^1, u^1) - f(t, x^2, u^2)| \leq L_K(|x^1 - x^2| + |u^1 - u^2|) \quad \text{and}$$
$$|g(t, s, x^1) - g(t, s, x^2)| \leq L_K|x^1 - x^2| \quad \text{for all } t \in I, \; s \in I$$

(iii) *For each $i = 1, ..., \nu$, each $t \in I$ and each $(x_1, ..., x_i)^T$ in $\mathbf{R}^{m_1 + \cdots + m_i}$, we have $f_i(t, x_1, ..., x_i, \mathbf{R}^{m_{i+1}}) = \mathbf{R}^{m_i}$.*

Given $x^0 \in \mathbf{R}^n$, and $u(\cdot) \in L_\infty(I; \mathbf{R}^m)$, let $t \mapsto x(t, x^0, u(\cdot))$ denote the trajectory, of (2), defined by this control $u(\cdot)$ and by the initial condition $x(t_0) = x^0$ on the maximal interval $J \subset I$ of the existence of the solution. As in [12], we say that a system of the Volterra integro-differential equations is globally controllable in time $I = [t_0, T]$ in class $C^\mu(I; \mathbf{R}^m)$ $(\mu \geq 0)$, iff for each initial state $x^0 \in \mathbf{R}^n$ and each terminal state $x^T \in \mathbf{R}^n$, there is a control $u(\cdot)$ in $C^\mu(I; \mathbf{R}^m)$ which "steers x^0 into x^T w.r.t. the system", i.e., the trajectory $x(\cdot)$ of the system with this control $u(\cdot)$ such that $x(t_0) = x^0$ is well-defined on I and satisfies $x(T) = x^T$.

Following [12] and [13], we also consider a perturbation of system (2) of the form

$$\dot{x}(t) = f(t, x(t), u(t)) + h(t, x(t), u(t)) + \int_{t_0}^{t} g(t, s, x(s))ds +$$
$$+ \int_{t_0}^{t} r(t, s, x(s), u(s))ds, \quad t \in I = [t_0, T] \quad (4)$$

where functions h and r satisfy the conditions:

(iv) $h \in C(I \times \mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$, $r \in C(I^2 \times \mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$, *and for each compact set $Q \subset \mathbf{R}^n \times \mathbf{R}^m$, there exists $L_Q > 0$ such that, for all $(t, s) \in I^2$, $(x^1, u^1) \in Q$, $(x^2, u^2) \in Q$, we have:*

$$|h(t, x^1, u^1) - h(t, x^2, u^2)| \leq L_Q(|x^1 - x^2| + |u^1 - u^2|),$$
$$|r(t, s, x^1, u^1) - r(t, s, x^2, u^2)| \leq L_Q(|x^1 - x^2| + |u^1 - u^2|),$$

(v) *There exists $H > 0$ such that h and r satisfy the inequalities $|h(t, x, u)| \leq H$ and $|r(t, s, x, u)| \leq H$ for all $(t, s, x, u) \in I^2 \times \mathbf{R}^n \times \mathbf{R}^m$.*

3. MAIN RESULTS

Theorem 3.1 *Suppose that system (2) has the form (3) and satisfies conditions (i),(ii),(iii). Then system (2) is globally controllable in class $C^\infty(I; \mathbf{R}^m)$.*

Theorem 3.2 *Suppose that functions f and g have the form (3), satisfy (i), (iii), and satisfy the* global *Lipschitz condition w.r.t. x and u, instead of (ii), i.e., there exists*

$L>0$ such that for each $(t,s)\in I^2$, each $(x^1,u^1)\in\mathbf{R}^n\times\mathbf{R}^m$ and each $(x^2,u^2)\in\mathbf{R}^n\times\mathbf{R}^m$ we have

$$|f(t,x^1,u^1)-f(t,x^2,u^2)|\le L(|x^1-x^2|+|u^1-u^2|),$$
$$|g(t,s,x^1)-g(t,s,x^2)|\le L|x^1-x^2|.$$

Suppose h and r satisfy (iv), (v). Then system (4) is globally controllable in time I by means of controls of class $C^\infty(I;\mathbf{R}^m)$.

Remark 3.1 Let us compare the results of [12] with our Theorems 3.1 and 3.2. First, in [12], functions f and g are required not only to be continuous but also to have all their partial derivatives, w.r.t. x and u, which are required to be continuous whereas we require (i) and (ii) only; the latter being the standard condition needed to guarantee the existence and the uniqueness of the solution of the "Cauchy problem" for the Volterra systems. Second, our system (2) is MIMO and furthermore x_i and u are vectors of different dimensions whereas, in [12], the system is SISO (i.e., x_i and u are scalar) or at least x_i and u should be of the same dimension (see Remark 3.1 from [12]). Third (and this is essential), our current Assumption (iii) is much more general than the corresponding Assumption (ii) (or (II), p.247) from [12]. In this sense, our current Theorem 3.1 and Theorem 3.2 generalize Theorem 3.3 and Theorem 3.2 from [12] respectively. However: firstly, in our case, function g has a bit more specific form than function g from [12] (g_i does not depend on x_{i+1} in the current paper); secondly, since we replace the assumption of C^1 smoothness with that of local Lipschitzness, we do not obtain stronger results on robustness (Theorem 3.1 from [12]).

Example 3.1. Consider the system given by

$$\begin{cases}\dot x_1(t)=(x_2(t)+x_1(t))|\sin x_2(t)|+\int\limits_0^t\sqrt{s^2x_1^2(s)+1}ds\\ \dot x_2(t)=u(t)|\cos u(t)|+\int\limits_0^t\sqrt{e^{ts}(x_1^2(s)+x_2^2(s))+1}ds,\end{cases}\tag{5}$$

$t\in[0,T]$. It is clear that systems (5) satisfies our Assumptions (i)-(iii) and therefore is globally controllable by Theorem 3.1. On the other hand, system (5) does not satisfy the Assumptions from [12] and the results of [12] are not applicable to system (5).

Remark 3.2 Note that, if $g=0$ in (2), then (2) is reduced to the class of the so-called "generalized triangular form" of ODE control systems considered in [13, 19, 18]. However, in the case of ODE, stronger results were obtained in these works: global robust controllability (Theorem 3.1 from [13]), global asymptotic stabilization by means of smooth controls (Theorem 2.1 from [19]), and global discontinuous stabilization in the sense of Clarke-Ledyaev-Sontag-Subbotin (Theorem 3.4 from [13]).

4. BACKSTEPPING IN THE NON-SMOOTH CASE

Let us reduce Theorem 3.1 and Theorem 3.2 to a backstepping process which can be compared with that from [13].

Let p be in $\{1,\dots,\nu\}$. Define $k:=m_1+...+m_p$ and consider the following k - dimensional control system

$$\dot y(t)=\varphi(t,y(t),v(t))+\int\limits_{t_0}^t\psi(t,s,y(s))ds,\ t\in I=[t_0,T]\tag{6}$$

where $y:=(x_1,...,x_p)^T\in\mathbf{R}^k=\mathbf{R}^{m_1+\dots+m_p}$ is the state, $v\in\mathbf{R}^{m_{p+1}}$ is the control and

$$\varphi(t,y,v)=\begin{pmatrix}f_1(t,x_1,x_2)\\ f_2(t,x_1,x_2,x_3)\\ \dots\\ f_p(t,x_1,...,x_p,v)\end{pmatrix}\tag{7}$$

$$\psi(t,s,y)=\begin{pmatrix}g_1(t,s,x_1)\\ g_2(t,s,x_1,x_2)\\ \dots\\ g_p(t,s,x_1,...,x_p)\end{pmatrix}\tag{8}$$

for all (t,y,v) in $I\times\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$. Given $y^0\in\mathbf{R}^k$, and $v(\cdot)\in L_\infty(I;\mathbf{R}^{m_{p+1}})$, let $t\mapsto y(t,y^0,v(\cdot))$ denote the trajectory, of (6), defined by the control $v(\cdot)$ and by the initial condition $y(t_0,y^0,v(\cdot))=y^0$ on the maximal interval $J\subset I$ of the existence of the solution. We reduce the proofs of Theorems 3.1 and 3.2 to the following Theorem.

Theorem 4.1 *Let p be in $\{1,...,\nu\}$. Suppose for each $y^0\in\mathbf{R}^k$ and each $\delta>0$, there is a family of functions $\{y(\xi,\cdot)=(x_1(\xi,\cdot),...,x_p(\xi,\cdot))\}_{\xi\in\mathbf{R}^k}$ such that:*

1) The map $\xi\mapsto y(\xi,\cdot)$ is of class $C(\mathbf{R}^k;C^1(I;\mathbf{R}^k))$

2) For each $\xi\in\mathbf{R}^k$ we have:

$$\dot x_i(\xi,t)=f_i(t,x_1(\xi,t),\dots,x_{i+1}(\xi,t))+$$
$$+\int\limits_{t_0}^t g_i(t,s,x_1(\xi,s),\dots,x_i(\xi,s))ds,\quad t\in I,\ \ 1\le i\le p-1;$$

(if $p=1$, then, by definition, the set of equalities is empty and, by definition, Condition 2) holds true)

3) $y(\xi,t_0)=y^0$ and $|y(\xi,T)-\xi|<\delta$ for all $\xi\in\mathbf{R}^k$

Then, for each $(y^0,y^0_{p+1})\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, and each $\varepsilon>0$, there exists a family of controls $\{\hat v_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ such that

4) The map $(\xi,\beta)\mapsto\hat v_{(\xi,\beta)}(\cdot)$ is of class $C(\mathbf{R}^k\times\mathbf{R}^{m_{p+1}};C^\infty(I;\mathbf{R}^{m_{p+1}}))$

5) For each $(\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$, we have $\hat v_{(\xi,\beta)}(T)=\beta$ and $\hat v_{(\xi,\beta)}(t_0)=y^0_{p+1}$.

6) $|y(T,y^0,\hat v_{(\xi,\beta)}(\cdot))-\xi|<\varepsilon$ for all $(\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}$.

Let us prove that Theorem 3.1 and Theorem 3.2 follow from Theorem 4.1. Indeed, suppose Theorem 4.1 holds true.

Suppose $p=1$ and $k=m_1$, and take an arbitrary $y^0_1\in\mathbf{R}^{m_1}$. Given an arbitrary $\delta>0$, find any family $\{y(\eta,\cdot)\}_{\eta\in\mathbf{R}^{m_1}}=\{x_1(\eta,\cdot)\}_{\xi\in\mathbf{R}^{m_1}}$ such that Conditions 1)-3) of Theorem 4.1 hold. Then, for p=1, we have: for every $\varepsilon>0$ and every $(y^0_1,y^0_2)\in\mathbf{R}^{m_1+m_2}$, there exists a family of controls $\{\hat v_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold with p=1.

Suppose p=2. Given any y^0=(y^0_1, y^0_2)∈$\mathbf{R}^{m_1+m_2}$, and any δ>0, define ε:=δ, and for this ε>0 find the family $\{\hat{v}_{(\eta,\beta)}(\cdot)\}_{(\eta,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ obtained at the previous step (with $p = 1$). From Conditions 4)-6) applied to $p = 1$ it follows that the family $\{y(\xi,\cdot)\}_{\xi=(\eta,\beta)\in\mathbf{R}^{m_1}\times\mathbf{R}^{m_2}}$ defined by

$$y(\eta,\beta,t) := (y(t, y^0_1, \hat{v}_{(\eta,\beta)}(\cdot)), \hat{v}_{(\eta,\beta)}(t)),$$
$$\text{for all} \quad t \in I, \quad \xi = (\eta,\beta) \in \mathbf{R}^{m_1} \times \mathbf{R}^{m_2}$$

satisfies the Conditions 1), 2), 3) of Theorem 4.1 with $p = 2$. Then we can apply Theorem 4.1 to $p = 2$, etc. Arguing by induction over $p = 1, \ldots, \nu$, we obtain for $p = \nu$: for each ε>0, each x^0∈$\mathbf{R}^n$, and each α=$y^0_{\nu+1}$∈$\mathbf{R}^{m_{\nu+1}}$ there exists a family of controls $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^n\times\mathbf{R}^{m_{\nu+1}}}$ such that Conditions 4), 5), 6) of Theorem 4.1 hold for $p = \nu$. Fix an arbitrary $\beta \in \mathbf{R}^{m_{\nu+1}}$ and define the family of controls $\{u_\xi(\cdot)\}_{\xi\in\mathbf{R}^n}$ as follows: $u_\xi(t) := \hat{v}_{(\xi,\beta)}(t)$ for all t∈I, ξ∈$\mathbf{R}^n$. Then $\{u_\eta(\cdot)\}_{\eta\in\mathbf{R}^n}$ satisfies the conditions:

(a) $\xi \mapsto u_\xi(\cdot)$ is of class $C(\mathbf{R}^n; C^\infty(I; \mathbf{R}^{m_{\nu+1}}))$

(b) For each $\xi \in \mathbf{R}^n$, the trajectory $t \mapsto x(t, x^0, u_\xi(\cdot))$ is well-defined and $|x(T, x^0, u_\xi(\cdot)) - \xi| < \varepsilon$.

Given any ε>0, an arbitrary x^0∈$\mathbf{R}^n$, and an arbitrary x^T∈$\mathbf{R}^n$, let $\{u_\xi(\cdot)\}_{\xi\in\mathbf{R}^n}$ be a family of controls such that (a), (b) hold. By conditions (a),(b) the map $\xi \mapsto \xi - x(T, x^0, u_\xi(\cdot)) + x^T$ is well-defined and of class $C(\mathbf{R}^n; \mathbf{R}^n)$. From condition (b), it follows that this continuous function maps the compact convex set $\overline{B_\varepsilon(x^T)}$ into $\overline{B_\varepsilon(x^T)}$. Then, by the Brouwer fixed-point theorem, there exists $\xi^* \in \overline{B_\varepsilon(x^T)} \subset \mathbf{R}^n$ such that ξ^*=ξ^*−$x(T, x^0, u_{\xi^*}(\cdot))$+$x^T$, i.e., $x(T, x^0, u_{\xi^*}(\cdot)) = x^T$. Thus, for every x^0∈$\mathbf{R}^n$, and every x^T∈$\mathbf{R}^n$, there is a control $u_{\xi^*}(\cdot) \in C^\infty(I; \mathbf{R}^{m_{\nu+1}})$ such that $x^T = x(T, x^0, u_{\xi^*}(\cdot))$, i.e., Theorem 3.1 follows from Theorem 4.1.

The proof of Theorem 3.2 is similar to this argument: having constructed the familty $\{u_\xi(\cdot)\}_{\xi\in\mathbf{R}^n}$ such that conditions (a),(b) hold for each $\xi \in \mathbf{R}^n$, by $x(\xi,\cdot)$ denote the trajectory, of (4), defined by the control $u_\xi(\cdot)$ and by the initial condition $x(\xi, t_0)$=x^0. Using the Gronwall-Bellmann lemma, we easily obtain that t↦$x(\xi,t)$ is well-defined for all t∈I, ξ∈$\mathbf{R}^n$ and there exists $D > 0$ such that $|x(\xi,t) - x(t, x^0, u_\xi(\cdot))| \le D$ for all t∈I and ξ∈$\mathbf{R}^n$, and therefore, by condition (b), we obtain: $|x(\xi,T) - \xi| \le D + \varepsilon$ for all ξ∈$\mathbf{R}^n$. Taking an arbitrary x^T∈$\mathbf{R}^n$ and applying the Brouwer fixed-point theorem to the map ξ↦ξ−$x(\xi,T)$+x^T, which maps the closed ball $\overline{B_{D+\varepsilon}(x^T)}$ into $\overline{B_{D+\varepsilon}(x^T)}$, we obtain the existence of $\xi^* \in \overline{B_{D+\varepsilon}(x^T)} \subset \mathbf{R}^n$ such that $x^T = x(\xi^*, T)$, which means that the control $u_{\xi^*}(\cdot)$∈$C^\infty(I; \mathbf{R}^m)$ steers x^0 into x^T in time I w.r.t. system (4). Since x^0 and x^T are chosen arbitrarely, the proof of Theorem 3.2 is complete.

5. PROOF OF THEOREM 4.1

Fix an arbitrary p in $\{1, \ldots, \nu\}$ an arbitrary $(y^0, y^0_{p+1}) \in \mathbf{R}^k \times \mathbf{R}^{m_{p+1}}$, and an arbitrary $\varepsilon > 0$. Define $\delta := \frac{\varepsilon}{4}$ and assume that $\{y(\xi,\cdot)\}_{\xi\in\mathbf{R}^k}$ satisfies Assumptions 1)-3) of Theorem 4.1

To prove Theorem 4.1, we change the approach from [12] and [13] as follows. Along with system (6), we consider the following k - dimensional control system of the Volterra equations

$$\begin{cases} \dot{x}_i(t) = f_i(t, x_1(t), ..., x_{i+1}(t)) + \\ + \displaystyle\int_{t_0}^{t} g_i(t, s, x_1(s), ..., x_i(s))ds, \quad i=1, ..., p-1; \\ \dot{x}_p(t) = \omega(t) + \displaystyle\int_{t_0}^{t} g_p(t, s, x_1(s), ..., x_p(s))ds; \end{cases} \quad t \in I \quad (9)$$

with states y=$(x_1, ..., x_p)^T \in \mathbf{R}^k$ and controls ω∈$\mathbf{R}^{m_p}$. Given y∈$\mathbf{R}^k$, and $\omega(\cdot)$∈$L_\infty(I; \mathbf{R}^{m_p})$, let $t \mapsto z(t, y, \omega(\cdot))$ denote the trajectory, of (9), defined by the control $\omega(\cdot)$ and by the initial condition $z(t_0, y, \omega(\cdot)) = y$ on some maximal interval $J \subset I$ of the existence of the solution.

For all ξ∈$\mathbf{R}^k$, define

$$\omega(\xi,t) = \dot{x}_p(\xi,t) - \int_{t_0}^{t} g_p(t, s, x_1(\xi,s), ..., x_p(\xi,s))ds, \ t \in I. \quad (10)$$

Then

$$y(\xi,t) = z(t, y^0, \omega(\xi,\cdot)) \text{ for all } \quad t \in I, \ \ \xi \in \mathbf{R}^k. \quad (11)$$

Then, using the Gronwall-Bellmann lemma, we get the existence of $\delta(\cdot)$ in $C(\mathbf{R}^k;]0, +\infty[)$ such that, for each ξ∈$\mathbf{R}^k$ and each $\omega(\cdot) \in L_\infty(I; \mathbf{R}^{m_p})$, we have:

$$\forall \, t \in I \quad |z(t, y^0, \omega(\cdot)) - y(\xi,t)| < \delta,$$
$$\text{whenever} \quad \| \omega(\cdot) - \omega(\xi,\cdot)\|_{L_\infty(I;\mathbf{R}^{m_p})} < \delta(\xi). \quad (12)$$

In order to complete the proof of Theorem 4.1, it suffices to prove the following Statement, which is similar to Lemma 5.1 from [13].

Statement 5.1. *Assume that* $\{y(\xi,\cdot)\}_{\xi\in\mathbf{R}^k}$ *is a family such that Conditions 1)-3) of Theorem 4.1 hold. Then, for system (6), there exist functions* $M(\cdot)$∈$C(\mathbf{R}^k;]0, +\infty[)$ *and a family* $\{u(\xi,\cdot)\}_{\xi\in\mathbf{R}^k}$ *of controls defined on I such that:*

1) For each ξ∈$\mathbf{R}^k$, *the control* $u(\xi,\cdot)$ *is a piecewise constant function on I and the map* ξ↦$u(\xi,\cdot)$ *is of class* $C(\mathbf{R}^k; L_1(I; \mathbf{R}^{m_{p+1}}))$.

2) For each ξ∈$\mathbf{R}^k$, *the trajectory* t↦$y(t, y^0, u(\xi,\cdot))$ *is defined for all* t∈I, *and for each* ξ∈$\mathbf{R}^k$ *we have*

$$|\omega(\xi,t) - f_p(t, y(t, y^0, u(\xi,\cdot)), u(\xi,t))| < \delta(\xi), \ \ t \in I$$

3) For each ξ∈$\mathbf{R}^k$, *we have:* $\| u(\xi,\cdot)\|_{L_\infty(I;\mathbf{R}^{m_{p+1}})} \le M(\xi)$.

Indeed, if Statement 5.1 is proved, then, combining (10), (11), (12) with the form of the dynamics of (6),(9), we get

$$|y(t, y^0, u(\xi,\cdot)) - y(\xi,t)| < \delta \quad \text{for all } \ t \in I, \ \xi \in \mathbf{R}^k. \quad (13)$$

Using partitions of unity and arguing as in [12], [13], we get the existence of a family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ of controls such that Conditions 4) and 5) of Theorem 4.1 hold and such that for each $(\xi,\beta) \in \mathbf{R}^k \times \mathbf{R}^{m_{p+1}}$ we have

$$|y(t,y^0,\hat{v}_{(\xi,\beta)}(\cdot)) - y(t,y^0,u(\xi,\cdot))| < \delta \text{ for all } t \in I, \quad (14)$$

($t \mapsto y(t,y^0,\hat{v}_{(\xi,\beta)}(\cdot))$ being defined on I for all (ξ,β) in $\mathbf{R}^k \times \mathbf{R}^{m_{p+1}}$). Since $\delta = \frac{\varepsilon}{4}$, from (13), (14) and from Assumption 3) of Theorem 4.1 it follows that the family $\{\hat{v}_{(\xi,\beta)}(\cdot)\}_{(\xi,\beta)\in\mathbf{R}^k\times\mathbf{R}^{m_{p+1}}}$ also satisfies Condition 6) of Theorem 4.1. This completes the proof of Theorem 4.1.

Due to space limits, we must omit the proof of Statement 5.1, which is similar to Lemma 5.1 from [13] modulo to the following distinctions:

($\star$) In the current paper, we deal with the Volterra systems whereas [13] is devoted to the case of ODE.

($\star\star$) In the current work, the parameter ξ characterizes the terminal state the system should be steered to starting from the initial point $y^0 \in \mathbf{R}^k$. In [13], the construction starts with the initial condition $z(\xi,T) = \xi$ given at the *terminal* instant T, and then the control strategy is adjusted inductively ([13], Lemma 6.1) while time is decreasing (from $t = T$ until the initial instant $t = t_1$) in order to reach a certain small neighborhood of the initial state. However, for the Volterra systems, such an invertion of time is not possible in general (and one cannot consider the Cauchy initial condition at terminal instant T). Therefore the direct repetition of the argument from [13], Section 6 would not suit.

($\star\star\star$) In the current work, we consider the non-smooth case (the right-hand side of (2) satisfies the local Lipschitz condition only).

It appears however that the proof of Lemma 5.1 from [13] can be changed: ($\star\star\star$) is not essential as the construction from [13], Section 6 does not refer to any C^1 - smoothness and local Lipschitzness does suffice; as to ($\star$), ($\star\star$) all the Section 6 from [13] can be revised with the correspoding changes. Thus, the proofs of Theorem 4.1, Theorem 3.1 and Theorem 3.2 are complete.

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