Further Remarks on Global Stabilization of Generalized Triangular Systems

Sergey Dashkovskiy and Svyatoslav S. Pavlichkov

***Abstract*— We remove the assumption on C^ν - smoothness from the main theorem in work [7] and show how to modify the argument from [7] in order to obtain the same result on global asymptotical stabilization when the dynamics satisfies the local Lipschitz condition in general and is of class C^1 around the equilibrium only.**

I. INTRODUCTION

THis work is motivated by the issue of global backstepping design and constructing global asymptotic stabilizers for the case of singular input-output "interconnections", when a control system has a triangular form ([4], [5])

$$\begin{cases} \dot{x}_i = f_i(t, x_1, ..., x_{i+1}), \quad i = 1, \ldots, \nu - 1; \\ \dot{x}_\nu = f_\nu(t, x_1, ..., x_\nu, u), \end{cases}$$

but is not feedback lineairzable, which means (see [3]) that the condition $|\frac{\partial f_i}{\partial x_{i+1}}| \neq 0$ does not necessarily hold true. This can occur even in quite simple cases, for instance, if one deals with polynomial forms. In work [1], the problem of feedback triangulation was investigated under the assumption that the set of regular points is open and dense in the state space. Furthermore, in [2], the problem of local stabilization was investigated under the assumption that one of the characteristic numbers $\frac{\partial^k f_i(x^*)}{\partial x_{i+1}^k}$, $k = 1, 2, \ldots$ is different from zero at the equilibrum point for each $i = 1, \ldots, \nu$, and, in [8] global stabilization was obtained when $f_i(t, x_1, ..., x_i, \cdot)$ are surjections and satisfy some additional "growth conditions" - see A3, (i), (ii) and (iii). This led to the concept of the so-called "generalized triangular form", when the only assumption is that $f_i(t, x_1, ..., x_i, \cdot)$ is a surjection (and x_i, u are vectors not necessarily of the same dimension). For this general case, the problem of global robust controllability was completely solved in [6] and the global asymptotic stabilization was obtained in [7].

In the current paper, we want to explain how to remove some assumptions on smoothness of f_i imposed in [7]. Let us remark that in many applications the right-hand side appears to be non-smooth. On the other hand, the C^ν - smoothness was essential even in the classical backstepping and feedback linearization theory, therefore, when dealing with the generalized triangular form it is an interesting problem to remove this assumption on smoothness.

This work was supported in part by the German Research Foundation (DFG) as part of the Collaborative Research Center 637 "Autonomous Cooperating Logistic Processes: A Paradigm Shift and its Limitations" (SFB 637)

Sergey Dashkovskiy is with with Center of Industrial Mathematics, University of Bremen, 28359 Bremen, Germany dsn@math.uni-bremen.de

Svyatoslav S. Pavlichkov is with Center of Industrial Mathematics, University of Bremen, 28359 Bremen, Germany s_s_pavlichkov@yahoo.com

II. MAIN RESULT

We consider the following system

$$\dot{x} = f(t, x, u), \tag{1}$$

where $u \in \mathbf{R}^m = \mathbf{R}^{m_{\nu+1}}$ is the control, $x = [x_1, ..., x_\nu]^T \in \mathbf{R}^n$ are the states with $x_i \in \mathbf{R}^{m_i}$, $m_i \leq m_{i+1}$, $n = m_1 + \ldots + m_\nu$, function f has the triangular form

$$f(t, x, u) = \begin{bmatrix} f_1(t, x_1, x_2) \\ f_2(t, x_1, x_2, x_3) \\ \ldots \\ f_\nu(t, x_1, ..., x_\nu, u) \end{bmatrix} \tag{2}$$

with $f_i(t, x_1, ..., x_{i+1}) \in \mathbf{R}^{m_i}$, and the system satisfies the following assumptions:

(A1) (a) $f \in C(\mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m; \mathbf{R}^n)$, and $f(t + T, x, u) = f(t, x, u)$ for all $[t, x, u] \in \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$ with some $T > 0$;
(b) f satisfies the local Lipschitz condition w.r.t. the states and controls, i.e., for every compact set $K \subset \mathbf{R}^n \times \mathbf{R}^m$ there is $L_K > 0$ such that, for every $(x^1, u^1) \in K$ and every $(x_2, u_2) \in K$ we obtain

$$|f(t, x^1, u^1) - f(t, x^2, u^2)| \leq L_K(|x^1 - x^2| + |u^1 - u^2|)$$
$$\text{for all} \quad t \in [0, T]$$

(c) f is of class $C^1(E; \mathbf{R}^n)$, where $E \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$ is some small open neighborhood of the set $\mathbf{R} \times \{[0, 0]\} \subset \mathbf{R} \times \mathbf{R}^n \times \mathbf{R}^m$.

(A2) $f_i(t, x_1, ..., x_i, \mathbf{R}^{m_{i+1}}) = \mathbf{R}^{m_i}$ for each $[t, x_1, ..., x_i] \in [0, T] \times \mathbf{R}^{m_1} \times \ldots \times \mathbf{R}^{m_i}$, $i = 1, ..., \nu$.

(A3) there exist $x_i^* \in \mathbf{R}^{m_i}$, $1 \leq i \leq \nu$, and $u^* = x_{\nu+1}^*$ in $\mathbf{R}^m$ such that rank $\frac{\partial f_i}{\partial x_{i+1}}(t, x_1^*, ..., x_{i+1}^*) = m_i$ for every $t \in [0, T]$, $i = 1, ..., \nu$, and such that $f(t, x^*, u^*) = 0$ for all $t \in [0, T]$.

The goal of the current paper is to prove that the main result of [7] still holds true if we replace Assumtions A1-A3 from [7] with the above Assumptions (A1)-(A3) (the difference is in Assumption (A1): in work [7], it is assumed that f is of class C^ν (instead of the current (A1(b),(c))) and T -periodic in time, which is much more restrictive than our current Assumption (A1). More specifically, we prove the following Theorem:

Theorem 1 *Suppose that system (1) satisfies the above conditions (A1)-(A3). Then system (1) is globally asymptotically stabilizable by means of a C^∞ time-varying T-periodic*

ISBN 978-963-311-370-7

feedback law, i.e., there exists a feedback law $u(t,x)$ of class $C^\infty(\mathbf{R}\times\mathbf{R}^n;\mathbf{R}^m)$ such that $u(t+T,x)=u(t,x)$ for all $[t,x]\in\mathbf{R}\times\mathbf{R}^n$ and $u(t,x^*)=u^*$ for all $t\in\mathbf{R}$, and such that the equilibrium point x^* is globally asymptotically stable for system (1) with $u=u(t,x)$.

Finally, let us compare our class with that considered in [9]. The latter (in particular) satisfies Condition 1.1 (A.1) from [9] (see also inclusion (1.2) from [9]), which automatically implies our Assumption (A2). Furthermore our class in not necessarily SISO (x_i and u can be vectors) and our dynamics is not necessarily of class C^∞ as in [9]. In this sense, our generalized TF defined by our Assumptions (A1), (A2) is an extension of the class considered in [9] (as well as an extension of the classes considered in [1], [2], [4], [5], [8]). On the other hand, our equilibrium point the systems should be stabilized to is assumed to be regular (see our Assumption (A3)), while in [9] only $\frac{\partial^{q_i} f_i}{\partial x_{i+1}^{q_i}}\neq 0$ with some odd q_i is required.

III. GLOBAL BACKSTEPPING IN THE SINGULAR CASE

In this paper, we keep all the notation from work [7]. Following [7], we take any p in $\{0,\dots,\nu-1\}$, and put $k:=m_1+\ldots+m_p$, if $p\geq 1$, and $k=0$ if $p=0$. Similarly for $y_0\in\mathbf{R}^{k+m_{p+1}}$, $\omega_0\in\mathbf{R}^{m_{p+1}}$, and $r>0$, we denote by $B_r(y_0)$ the open ball

$$B_r(y_0):=\{y\in\mathbf{R}^{k+m_{p+1}}\mid\ |y-y_0|<r\};$$

and by $\overline{B}_r(y_0)$ the closed ball

$$\overline{B}_r(y_0):=\{y\in\mathbf{R}^{k+m_{p+1}}\mid\ |y-y_0|\leq r\}.$$

By $\|\cdot\|$ we denote the matrix norm in $\mathbf{R}^{M\times N}$ with any finite M and N (it will be clear from the context which dimensions M and N are considered). Then, following [7], we consider a control system

$$\dot z=g(t,z,z_{p+1}),\qquad t\in\mathbf{R}\tag{3}$$

with states $z=[z_1,\dots,z_p]^T\in\mathbf{R}^k$, where $z_i\in\mathbf{R}^{m_i}$, $i=1,\dots,p$, $p\geq 1$ and controls $z_{p+1}\in\mathbf{R}^{m_{p+1}}$ and its dynamical extension

$$\dot y=\psi(t,y,v),\qquad t\in\mathbf{R}\tag{4}$$

with states $y=[z,z_{p+1}]^T\in\mathbf{R}^{k+m_{p+1}}$ and controls $v\in\mathbf{R}^{m_{p+2}}$ where $\psi(t,y,v)=\psi(t,z,z_{p+1},v)$ has the form

$$\psi(t,y,v)=[g(t,y),\ g_{p+1}(t,y,v)]^T$$
$$\text{for all}\quad [t,y,v]\in\mathbf{R}\times\mathbf{R}^{k+m_{p+1}}\times\mathbf{R}^{m_{p+2}}\tag{5}$$

with $g_{p+1}\in\mathbf{R}^{m_{p+1}}$.

As well as in [7], for the case p=0, we say, by definition that (3) is empty and $y=z_{p+1}=z_1$; $\psi(t,y,v)=g_{p+1}(t,y,v)=g_1(t,z_1,v)$ with $v\in\mathbf{R}^{m_2}$ and that $\dot z_1=g_1(t,z_1,z_2)$ with states $z_1=y$ and controls $z_2=v$ is the extension of the empty system (3).

In contrast to [7], we do not require the dynamics to be smooth, it suffices to require the dynamics to satisfy the local Lipschitz condition w.r.t. states and controls and to be of class C^1 in some small neighborhood of the equilibrium only.

More specifically, we assume that:

B1: (a) $\psi(t,y,v)$ is a $T-$ periodic function with $T>0$ (i.e., there is $T>0$ such that $\psi(t+T,y,v)=\psi(t,y,v)$ for all $[t,y,v]$ in $\mathbf{R}\times\mathbf{R}^{k+m_{p+1}+m_{p+2}}$); and $\psi(t,y,v)$ is of class $C(\mathbf{R}\times\mathbf{R}^{k+m_{p+1}+m_{p+2}};\mathbf{R}^{k+m_{p+1}})$,
(b) ψ satisfies the local Lipschitz condition w.r.t. y and v, i.e., for each compact set $K\subset\mathbf{R}\times\mathbf{R}^{k+m_{p+1}+m_{p+2}}$ there exists $L_K>0$ such that

$$|\psi(t,y_1,v_1)-\psi(t,y_2,v_2)|\leq L_K(|y_1-y_2|+|v_1-v_2|)$$
$$\text{for all}\quad [t,y_1,v_1]\in K\ [t,y_2,v_2]\in K$$

(c) ψ is of class $C^1(D;\mathbf{R}^{k+m_{p+1}})$, where $D\subset\mathbf{R}\times\mathbf{R}^{k+m_{p+1}+m_{p+2}}$ is some open neighborhood of the set $\mathbf{R}\times\{[0,0,0]\}\subset\mathbf{R}\times\mathbf{R}^{k+m_{p+1}+m_{p+2}}$.

B2: $g_{p+1}(t,y,\mathbf{R}^{m_{p+2}})=\mathbf{R}^{m_{p+1}}$ for every $[t,y]\in[0,T]\times\mathbf{R}^{k+m_{p+1}}$.

B3: For every $t\in\mathbf{R}$, we have: $\psi(t,0,0)=0$; and $\operatorname{rank}\frac{\partial g_{p+1}}{\partial v}(t,0,0)=m_{p+1}$.

(note that, since ψ is T - periodic, without loss of generality it can be assumed that $D=D_{\hat r}:=\mathbf{R}\times\{[y,v]\in\mathbf{R}^{k+m_{p+1}+m_{p+2}}\mid |y|+|v|<\hat r\}$ in Condition B1(b))

Following [7], we consider the following Lyapunov functions

$$V_p(z):=\langle z,z\rangle \text{ and } V_{p+1}(y):=\langle y,y\rangle=\langle z,z\rangle+\langle z_{p+1},z_{p+1}\rangle$$

for systems (3) and (4) respectively.

Our modification of the backstepping procedure proposed in [7] is as follows:

Theorem 2. *Let systems (3) and (4) satisfy Assumptions B1-B3. Assume that for $\lambda>0$ there exist sequences $\{r_q\}_{q=2}^{+\infty}\subset\mathbf{R}$ and $\{\rho_q\}_{q=1}^{+\infty}\subset\mathbf{R}$ such that $0<\rho_q<r_{q+1}<\rho_{q+1}$, for all $q\in\mathbf{N}$; with $r_q\to+\infty$, $\rho_q\to+\infty$ as $q\to\infty$, and the following conditions hold*

C1: $\frac{\partial V_p(z)}{\partial z}g(t,z,0)\leq-\lambda V_p(z)$ *whenever* $|z|^2<r_2^2$, $z\in\mathbf{R}^k$, $t\in[0,T]$.

C2: *For every $z_0\in\mathbf{R}^k$, and every $t_0\in[0,T]$ if $|z_0|^2\leq r_{q+2}^2$ with some $q\in\mathbf{N}$ then*

$$|z(t,t_0,z^0,0)|^2\leq\rho_{q+2}^2-\frac{t-t_0}{T}(\rho_{q+2}^2-\rho_q^2),$$
$$\textit{for all}\quad t\in[t_0,t_0+T].$$

Then, there exist $q_0\geq 0$ ($q_0\in\mathbf{Z}$), positive real numbers r_1, r_0, ..., r_{-q_0}, a sequence of positive real numbers $\{R_q\}_{q=-q_0-1}^{\infty}$ and a feedback control $v(\cdot,\cdot)$ of class $C^\infty(\mathbf{R}\times\mathbf{R}^{k+m_{p+1}};\mathbf{R}^{m_{p+2}})$ such that $0<R_q<r_{q+1}<R_{q+1}$, for all $q\geq-q_0-1$, $q\in\mathbf{Z}$ and such that

(i) *$v(T+t,y)=v(t,y)$ for all $[t,y]$ in $\mathbf{R}\times\mathbf{R}^{k+m_{p+1}}$, and $v(t,0)=0\in\mathbf{R}^{m_{p+2}}$ for all $t\in\mathbf{R}$.*

(ii) *For every $t\in\mathbf{R}$, and every $y=[z,z_{p+1}]^T$ in $\overline{B}_{r_{-q_0}}(0)$, we have*

$$\frac{\partial V_{p+1}(y)}{\partial y}\psi(t,y,v(t,y))\leq-\frac{\lambda}{2}V_{p+1}(y)$$

(iii) *For every* $y_0 \in \mathbf{R}^{k+m_{p+1}}$, *and every* $t_0 \in \mathbf{R}$ *if* $|y_0|^2 \le r_{q+2}^2$ *with some* $q \ge -q_0 - 1$, $q \in \mathbf{Z}$ *then*

$$|y(t, t_0, y^0, v(\cdot, \cdot))|^2 \le R_{q+2}^2 - \frac{t - t_0}{T}(R_{q+2}^2 - R_q^2),$$

$$\textit{for all} \quad t \in [t_0, t_0{+}T]$$

As well as in [7], if p=0, *i.e., system (3) is empty, then we say that for any* $\lambda > 0$ *C1, C2 hold by definition, and the Theorem states that, for the corresponding extension (4), there is a control* $v(\cdot, \cdot)$ *such that Conditions (i), (ii) and (iii) hold.*

It is straightforward that Theorem 1 follows from Theorem 2. Without loss of generality we assume that $x^* = 0$, $u^* = 0$. Take any $\lambda > 0$. Then, for $p = 0$, Theorem 2 yields the existence of a smooth T-periodic feedback $x_2 = \alpha(t, x_1)$ which stabilizes globally the system $\dot{x}_1 = f_1(t, x_1, x_2)$. Then we put $z_1 := x_1$, $z_2 := x_2 - \alpha(t, x_1)$, and $g_1(t, z_1, z_2) := f_1(t, z_1, z_2 + \alpha(t, z_1))$, and apply Theorem 2 with $p = 1$. Then, applying Theorem 2 from $p = 0, 1, \ldots, \nu - 1$, we obtain Theorem 1.

IV. PROOF OF THEOREM 2

Let us show how to modify Step 1 of the proof of Theorem 3.1 from [7] in order to prove our Theorem 2. We first prove Theorem 2 for $p \ge 1$ and then we explain how to modify the argument for $p = 0$ (it becomes simpler in comparison with the case $p \ge 1$).

Suppose conditions C1, C2 hold with some $\lambda > 0$.

Let us prove the existence of $r \in]0, \rho_1[$ and $\beta(\cdot, \cdot)$ of class $C^\infty(\mathbf{R} \times \overline{B}_{2r}(0); \mathbf{R}^{m_{p+2}})$ such that

$$\beta(t, 0) = 0;\ \beta(t{+}T, y) = \beta(t, y) \ \text{ for all } t{\in}\mathbf{R},\ y{\in}\mathbf{R}^{k+m_{p+1}} \tag{6}$$

and such that

$$\frac{\partial V_{p+1}(y)}{\partial y}\psi(t, y, \beta(t, y)) \le -\frac{\lambda}{2}V_{p+1}(y)$$

$$\text{for all} \ \ y = [z, z_{p+1}] \in \overline{B}_{2r}(0), \ \ t \in \mathbf{R},$$

i.e.,

$$2\langle z, g(t, z, z_{p+1})\rangle{+}2\langle z_{p+1}, g_{p+1}(t, z, z_{p+1}, \beta(t, z, z_{p+1}))\rangle \le$$

$$\le -\frac{\lambda}{2}\langle z, z\rangle - \frac{\lambda}{2}\langle z_{p+1}, z_{p+1}\rangle$$

$$\text{for all} \ \ y = [z, z_{p+1}] \in \overline{B}_{2r}(0), \ \ t \in \mathbf{R}. \tag{7}$$

Indeed, by condition C1 of Theorem 2, the derivative of V_{p+1} along the trajectories of (4) is as follows:

$$\begin{aligned}
\frac{dV_{p+1}}{dt} &= \frac{\partial V_{p+1}}{\partial y}\psi(t, y, v) \\
&= 2\langle z, g(t, z, z_{p+1})\rangle{+}2\langle z_{p+1}, g_{p+1}(t, z, z_{p+1}, v)\rangle \\
&= 2\langle z, g(t, z, 0)\rangle{+}2\langle z, g(t, z, z_{p+1}){-}g(t, z, 0)\rangle \\
&+ 2\langle z_{p+1}, g_{p+1}(t, z, z_{p+1}, v)\rangle \\
&\le -\lambda\langle z, z\rangle{+}2\langle z, \left[\int_0^1 \frac{\partial g(t, z, \sigma z_{p+1})}{\partial z_{p+1}}d\sigma\right] z_{p+1}\rangle \\
&+ 2\langle z_{p+1}, g_{p+1}(t, y, v)\rangle = -\lambda\langle z, z\rangle \\
&+ 2\langle z_{p+1}, g_{p+1}(t, y, v) + J^*(t, z, z_{p+1})z\rangle, \\
&\qquad \text{whenever } |z|^2{<}r_2^2,\ z{\in}\mathbf{R}^k,\ [t, z, z_{p+1}]{\in}D,
\end{aligned} \tag{8}$$

where

$$J(t, z, z_{p+1}) := \int_0^1 \frac{\partial g(t, z, \sigma z_{p+1})}{\partial z_{p+1}}d\sigma, \quad [t, z, z_{p+1}]{\in}D, \tag{9}$$

and the asterisk means "transposed".

For every $\mu > 0$, $t \in \mathbf{R}$, define:

$$\Phi_0^\mu(t) := [\overline{\phi}_0^\mu(t), \phi_0^\mu(t)], \tag{10}$$

where $\overline{\phi}_0^\mu(t)$ in $\mathbf{R}^{m_{p+2} \times k}$ and $\phi_0^\mu(t)$ in $\mathbf{R}^{m_{p+2} \times m_{p+1}}$ are given by

$$\overline{\phi}_0^\mu(t) := -\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* \times \left[\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v} \times \left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^*\right]^{-1} \left[\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial z} {+} J^*(t, 0, 0)\right] \tag{11}$$

$$\phi_0^\mu(t) := \left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* \Bigg[-\mu I_{m_{p+1} \times m_{p+1}} - \left[\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^*\right]^{-1} \times \frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial z_{p+1}}\Bigg] \tag{12}$$

Then $\Phi_0^\mu(\cdot)$ is T - periodic and of class $C(\mathbf{R}; \mathbf{R}^{m_{p+2} \times (k+m_{p+1})})$. Given $\mu{>}0$ and $\delta{>}0$, find any T - periodic function $\Phi_\delta^\mu(\cdot){=}[\overline{\phi}_\delta^\mu(t), \phi_\delta^\mu(t)]$ of class $C^\infty(\mathbf{R}; \mathbf{R}^{m_{p+2} \times (k+m_{p+1})})$ such that

$$\| \Phi_\delta^\mu(t) - \Phi_0^\mu(t) \| \le \delta \qquad \text{for all} \quad t \in \mathbf{R} \tag{13}$$

For every $[\xi, \xi_{p+1}]^T {\in} \mathbf{R}^{k+m_{p+1}}$, every $\mu{>}0$, every $\theta{\in}[0, 1]$, every $t \in \mathbf{R}$, and every $\delta \in \{0\} {\cup}]0, +\infty[$, define:

$$\beta_\delta(\mu, t, \xi, \xi_{p+1}) := \Phi_\delta^\mu(t)\left[\begin{array}{c}\xi \\ \xi_{p+1}\end{array}\right] = \overline{\phi}_\delta^\mu(t)\xi + \phi_\delta^\mu(t)\xi_{p+1} \tag{14}$$

$$\rho_\delta(\theta, \mu, t, \xi, \xi_{p+1}) := \langle \xi_{p+1}, g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1})) + {+}J^*(t, \theta\xi, \theta\xi_{p+1})\theta\xi\rangle \tag{15}$$

Next we define for each $\delta \in \{0\} \cup]0, +\infty[$:

$$H_\delta(\theta, \mu, t, \xi, \xi_{p+1}) := [\overline{h}_\delta(\theta, \mu, t, \xi, \xi_{p+1}), h_\delta(\theta, \mu, t, \xi, \xi_{p+1})] \tag{16}$$

where $\overline{h}_\delta$ and h_δ are given by

$$\overline{h}_\delta(\theta, \mu, t, \xi, \xi_{p+1}) := \frac{\partial g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1}))}{\partial z} + \frac{\partial g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1}))}{\partial v}\overline{\phi}^\mu_\delta(t) \tag{17}$$

$$h_\delta(\theta, \mu, t, \xi, \xi_{p+1}) := \frac{\partial g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1}))}{\partial z_{p+1}} + \frac{\partial g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1}))}{\partial v}\phi^\mu_\delta(t) \tag{18}$$

Note that, by (14),(16),(17),(18), we obtain:

$$\overline{h}_0(0, \mu, t, \xi, \xi_{p+1}) = -J^*(t, 0, 0); \tag{19}$$

$$h_0(0, \mu, t, \xi, \xi_{p+1}) = -\mu\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* \tag{20}$$

Furthermore, by (13), for each fixed $\mu > 0$, we obtain:

$$\forall \hat{\varepsilon}>0\ \exists \delta>0\ \exists r>0\ (\| h_\delta(\theta, \mu, t, \xi, \xi_{p+1}) - h_0(0, \mu, t, \xi, \xi_{p+1}) \| < \hat{\varepsilon} \text{ and } \| \overline{h}_\delta(\theta, \mu, t, \xi, \xi_{p+1}) + J^*(t, \xi, \xi_{p+1}) \| < \hat{\varepsilon}, \text{ whenever } [\xi, \xi_{p+1}] \in \overline{B}_{2r}(0),\ t \in \mathbf{R},\ \theta \in [0, 1]\big) \tag{21}$$

Fix an arbitrary $\mu > 0$ and $t \in \mathbf{R}$. Then, by the Lagrange mean-value theorem, for every $[\xi, \xi_{p+1}] \in \mathbf{R}^{k+m_{p+1}}$, there exists $\theta(\mu, t, \xi, \xi_{p+1}) \in [0, 1]$ such that

$$\begin{aligned}
&\rho_\delta(1, \mu, t, \xi, \xi_{p+1}) - \rho_\delta(0, \mu, t, \xi, \xi_{p+1}) = \frac{d}{d\theta}\langle \xi_{p+1}, g_{p+1}(t, \theta\xi, \theta\xi_{p+1}, \beta_\delta(\mu, t, \theta\xi, \theta\xi_{p+1}))\rangle|_{\theta=\theta(\mu,t,\xi,\xi_{p+1})}(1-0) \\
&+\langle \xi_{p+1}, J^*(t, \xi, \xi_{p+1})\xi\rangle = \\
&\langle \xi_{p+1}, \overline{h}_\delta(\theta(\mu, t, \xi, \xi_{p+1}), \mu, t, \xi, \xi_{p+1})\xi + h_\delta(\theta(\mu, t, \xi, \xi_{p+1}), \mu, t, \xi, \xi_{p+1})\xi_{p+1}\rangle \\
&+\langle \xi_{p+1}, J^*(t, \xi, \xi_{p+1})\xi\rangle = \\
&\left\langle -\mu\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* \xi_{p+1}, \xi_{p+1}\right\rangle + \langle \xi_{p+1}, (h_\delta(\theta(\mu, t, \xi, \xi_{p+1}), \mu, t, \xi, \xi_{p+1}) - \\
&-h_0(0, \mu, t, \xi, \xi_{p+1}))\xi_{p+1}\rangle + \langle \xi_{p+1}, (\overline{h}_\delta(\theta(\mu, t, \xi, \xi_{p+1}), \mu, t, \xi, \xi_{p+1}) + J^*(t, \xi, \xi_{p+1}))\xi\rangle
\end{aligned} \tag{22}$$

From Condition (B3) it follows that the symmetric matrix

$$\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^*$$

is positive definite for all $t \in [0, T]$ (i.e., for all $t \in \mathbf{R}$ due to the T - periodicity). Since it is a continuous and T - periodic function w.r.t. $t \in \mathbf{R}$, using the compactness of $[0, T]$, we obtain:

$$\exists \Lambda > 0 \quad \langle z_{p+1}, \left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* z_{p+1}\rangle \geq \Lambda\langle z_{p+1}, z_{p+1}\rangle \quad \text{for all } z_{p+1} \in \mathbf{R}^{m_{p+1}},\ t \in \mathbf{R} \tag{23}$$

Therefore there exists $\mu_0>0$ such that

$$\left\langle \xi_{p+1}, -\mu_0\left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right) \times \left(\frac{\partial g_{p+1}(t, 0, 0, 0)}{\partial v}\right)^* \xi_{p+1}\right\rangle \leq -\lambda\langle \xi_{p+1}, \xi_{p+1}\rangle \quad \text{for all } \xi_{p+1} \in \mathbf{R}^{m_{p+1}},\ t \in \mathbf{R} \tag{24}$$

Given an arbitrary (small) $\hat{\varepsilon} > 0$, find $r > 0$ and $\delta > 0$ such that

$$\| h_\delta(\theta, \mu_0, t, \xi, \xi_{p+1}) - h_0(0, \mu_0, t, \xi, \xi_{p+1}) \| < \hat{\varepsilon} \text{ and } \| \overline{h}_\delta(\theta, \mu_0, t, \xi, \xi_{p+1}) + J^*(t, \xi, \xi_{p+1}) \| < \hat{\varepsilon}, \text{ whenever } [\xi, \xi_{p+1}] \in \overline{B}_{2r}(0),\ \theta \in [0, 1],\ t \in \mathbf{R} \tag{25}$$

(This is possible because of (21)). Then, if $\hat{\varepsilon} > 0$ in (25) is small enough, we obtain:

$$\begin{aligned}
&\langle \xi_{p+1}, (h_\delta(\theta(\mu_0, t, \xi, \xi_{p+1}), \mu_0, t, \xi, \xi_{p+1}) - h_0(0, \mu_0, t, \xi, \xi_{p+1}))\xi_{p+1}\rangle \leq \frac{\lambda}{4}\langle \xi_{p+1}, \xi_{p+1}\rangle \text{ and} \\
&\langle \xi_{p+1}, (\overline{h}_\delta(\theta(\mu_0, t, \xi, \xi_{p+1}), \mu_0, t, \xi, \xi_{p+1}) + J^*(t, \xi, \xi_{p+1}))\xi\rangle \leq \frac{\lambda}{4}\langle \xi_{p+1}, \xi_{p+1}\rangle + \frac{\lambda}{4}\langle \xi, \xi\rangle \\
&\text{whenever } [\xi, \xi_{p+1}] \in \overline{B}_{2r}(0),\ t \in \mathbf{R}.
\end{aligned} \tag{26}$$

Combining (22), (24), and (26), we obtain:

$$\rho_\delta(1, \mu_0, t, \xi, \xi_{p+1}) - \rho_\delta(0, \mu_0, t, \xi, \xi_{p+1}) \leq -\frac{\lambda}{2}\langle \xi_{p+1}, \xi_{p+1}\rangle + \frac{\lambda}{4}\langle \xi, \xi\rangle \quad \text{whenever } [\xi, \xi_{p+1}]^T \in \overline{B}_{2r}(0),\ t \in \mathbf{R}. \tag{27}$$

Define $\beta(t, z, z_{p+1}) := \beta_\delta(\mu_0, t, z, z_{p+1})$. Combining (27) with (8) and (15), we obtain:

$$\frac{dV_{p+1}}{dt}|_{(p+1), v=\beta} \leq -\frac{\lambda}{2}V_{p+1}$$

This completes Step 1 from [7].

Remark. For p=0, (i.e., when (3) is empty, z is abscent, ψ=g_{p+1}=g_1 and satisfies B1-B3, z_{p+1}=z_1=y and C1,C2 hold by definition with any λ> 0), the above construction and the proof is similar, but becomes much simpler. For an arbitrary λ>0, we easily find a smooth, T - periodic feedback $\beta(t, y) = \beta(t, z_1)$ of $\mathbf{R}\times\mathbf{R}^{m_1}$ to $\mathbf{R}^{m_1}$ such that $\beta(t, 0)$=0 and

$$\langle y, \psi(t, y, \beta(t, y))\rangle = \langle z_{p+1}, g_{p+1}(t, z_{p+1}, \beta(t, z_{p+1}))\rangle \leq -\frac{\lambda}{2}\langle z_{p+1}, z_{p+1}\rangle = \langle y, y\rangle$$

in some neighborhood of $\mathbf{R}\times\{0\}$ by the repetition of the above argument with the following changes: $J(t,z,z_{p+1})$ and (9) are omitted; $\overline{\phi}_\delta^\mu(t)$ is empty and, in particular, $\Phi_0^\mu(t)=\phi_0^\mu(t)$ in (10); $\beta_\delta(\mu,t,\xi,\xi_{p+1})=\beta_\delta(\mu,t,\xi_{p+1})=\phi_\delta^\mu(t)\xi_{p+1}$ in (14);

$$\rho_\delta(\theta,\mu,t,\xi,\xi_{p+1}) = \rho_\delta(\theta,\mu,t,\xi_{p+1}) =$$
$$\langle \xi_{p+1}, g_{p+1}(t,\theta\xi_{p+1},\beta_\delta(\mu,t,\theta\xi_{p+1}))\rangle$$

in (15); $\overline{h}_\delta(\theta,\mu,t,\xi,\xi_{p+1})$ and (17), (19) are omitted and

$$H_\delta(\theta,\mu,t,\xi,\xi_{p+1}) = H_\delta(\theta,\mu,t,\xi_{p+1}) = h_\delta(\theta,\mu,t,\xi,\xi_{p+1})$$
$$= h_\delta(\theta,\mu,t,\xi_{p+1})$$

is given by (18); the second inequalities in (21) and in (25) are omitted; all the terms that contain $J^*(...)$ and $\overline{h}_\delta(...)$ $\overline{h}_0(...)$ in (22), (26) are omitted. Eventually, instead of (27), we obtain the inequality

$$\rho_\delta(1,\mu_0,t,\xi,\xi_{p+1}) - \rho_\delta(0,\mu_0,t,\xi,\xi_{p+1}) \le -\frac{\lambda}{2}\langle \xi_{p+1},\xi_{p+1}\rangle,$$
$$\text{whenever } \ [\xi,\xi_{p+1}]^T \in \overline{B}_{2r}(0), \ \ t\in\mathbf{R},$$

which completes the proof of Step 1 from [7] for $p=0$ with $\beta(t,z_{p+1}) := \beta_\delta(\mu_0,t,z,z_{p+1})$.

Thus, Step 1 from [7] is complete for any $p\geq 0$. In order to prove Theorem 2, it suffices to note that Steps 2-5 from [7] require only the property of the local existence and uniquiness as in the local Picard's existence theorem, i.e., the local Lipschitz condition w.r.t. $[y,v]$ and continuity w.r.t. $[t,y,v]$ will suffice. Therefore, arguing as in Steps 2-5 from [7], we complete the proofs of our Theorem 2 and Theorem 1.

REFERENCES

[1] S. Celikovsky, H. Nijmeijer, "Equivalence of nonlinear systems to triangular form: the singular case", *Systems and Control Letters*, vol. 27, pp. 135 – 144, 1996.

[2] S. Celikovsky, E. Aranda-Bricaire, "Constructive nonsmooth stabilization of triangular systems", *Systems and Control Letters*, vol. 36, 21 – 37, 1999.

[3] B. Jakubczyk, W. Respondek, "On linearization of control systems", *Bull. Acad. Sci. Polonaise Ser. Sci. Math.*, vol. 28, pp. 517-522, 1980.

[4] I. Kanellakopoulos , P. Kokotovic, A.S. Morse, Systematic design of adaptive controllers for feedback linearizable systems, *IEEE Trans. Automat. Control* **36** (1991) 1241–1253.

[5] V.I. Korobov, Controllability and stability of some nonlinear systems, *Differencial'nie uravneniya* **9** (1973) 614–619.

[6] V.I. Korobov, S.S. Pavlichkov, Global properties of the triangular systems in the singular case, *J. Math. Anal. Appl.* **309** (2008) 1426 – 1439.

[7] S.S. Pavlichkov, S.S. Ge, Global stabilization of the generalized MIMO triangular systems with singular input-output links, *IEEE Trans. Automat. Control* **54** (2009) 1794 – 1806.

[8] J. Tsinias, "Partial-state global stabilization for general triangular systems", *Syst. Contr. Lett.*, vol. 24, pp. 139–145, 1995.

[9] J. Tsinias, "Triangular systems: A global extension of the Coron-Praly Theorem on the existence of feedback-integrator stabilizers ", *European J. Control*, vol. 3, pp. 37–46, 1997.