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SMALL GAIN THEOREMS FOR LARGE SCALE SYSTEMS AND CONSTRUCTION OF ISS LYAPUNOV FUNCTIONS*

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Abstract. We consider interconnections of n nonlinear subsystems in the input-to-state stability (ISS) framework. For each subsystem an ISS Lyapunov function is given that treats the other subsystems as independent inputs. A gain matrix is used to encode the mutual dependencies of the systems in the network. Under a small gain assumption on the monotone operator induced by the gain matrix, a locally Lipschitz continuous ISS Lyapunov function is obtained constructively for the entire network by appropriately scaling the individual Lyapunov functions for the subsystems. The results are obtained in a general formulation of ISS; the cases of summation, maximization, and separation with respect to external gains are obtained as corollaries.

Key words. nonlinear systems, input-to-state stability, interconnected systems, large scale systems, Lipschitz ISS Lyapunov function, small gain condition

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1. Introduction.

In many applications large scale systems are obtained through the interconnection of a number of smaller components. The stability analysis of such interconnected systems may be a difficult task especially in the cases of a large number of subsystems, arbitrary interconnection topologies, and nonlinear subsystems.

One of the earliest tools in the stability analysis of feedback interconnections of nonlinear systems are small gain theorems. Such results have been obtained by many authors starting with [36]. These results are classically built on the notion of L^p gains; see [3] for a recent, very readable account of the developments in this area. While most small gain results for interconnected systems yield only sufficient conditions, in [3] it has been shown in a behavioral framework how the notion of gains can be modified so that the small gain condition is also necessary for robust stability.

Small gain theorems for large scale systems have been developed, e.g., in [26, 34, 23]. In [26] the notions of connective stability and stabilization are introduced for interconnections of linear systems using the concept of vector Lyapunov functions. In [23] stability conditions in terms of Lyapunov functions of subsystems have been derived. For the linear case characterizations of quadratic stability of large scale interconnections have been obtained in [16]. A common feature of these references is that the gains describing the interconnection are essentially linear. With the

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introduction of the concept of *input-to-state stability* (ISS) in [28], it has become a common approach to consider gains as nonlinear functions of the norm of the input. In this nonlinear case small gain results have been derived first for the interconnection of two systems in [18, 32]. A Lyapunov version of the same result is given in [17]. A general small gain condition for large scale ISS systems has been presented in [6]. Recently, such arguments have been used in the stability analysis of observers [1], in the stability analysis of decentralized model predictive control [22], and in the stability analysis of groups of autonomous vehicles.

During the revision of this paper it came to our attention that, following the first general small gain theorems for networks [21, 33, 5, 8, 7, 6], other generalizations of small gain results based on similar ideas have been obtained very recently using the maximization formulation of ISS: A generalized small gain theorem for output-Lagrange-input-to-output stable systems in network interconnections has been obtained in [19]. In this reference the authors study ISS in the maximization framework and conclude ISS from a small gain condition in the cycle formulation. It has been noted in [8] that in the maximum case the cycle condition is equivalent to the operator condition examined here. An extension of generalized small gain results to retarded functional differential equations based on the more general cycle condition and vector Lyapunov functions has recently been obtained in [20]. In this reference a construction of a Lyapunov function is shown which takes a different approach to the construction of an overall Lyapunov function. This construction depends vitally on the use of the maximum formulation of ISS.

In this paper we present sufficient conditions for the existence of an ISS Lyapunov function for a system obtained as the interconnection of many subsystems. The results are of interest in two ways. First, it is shown that a small gain condition is sufficient for ISS of the large scale system in the Lyapunov formulation. Second, an explicit formula for an overall Lyapunov function is given. As the dimensions of the subsystems are essentially lower than the dimension of their interconnection, finding Lyapunov functions for them may be an easier task than for the whole system.

Our approach is based on the notion of ISS introduced in [28] for nonlinear systems with inputs. A system is ISS if, roughly speaking, it is globally asymptotically stable in the absence of inputs (so-called 0-GAS) and if any trajectory eventually enters a ball centered at the equilibrium, which has a radius given by a monotone continuous function, the gain, of the size of the input (the so-called *asymptotic gain property*); cf. [31].

The concept of ISS turned out to be particularly well suited to the investigation of interconnections. For example, it is known that cascades of ISS systems are again ISS [28], and small gain results have been obtained. We briefly review the results of [18, 17] in order to explain the motivation for the approach of this paper. Both papers study a feedback interconnection of two ISS systems as represented in Figure 1.1.

The small gain condition in [18] is that the composition of the gain functions γ_{12}, γ_{21} is less than identity in a robust sense. We denote the composition of functions f, g by $\circ$; that is, $(f \circ g)(x) := f(g(x))$. The small gain condition then is that if on $(0, \infty)$ we have

$$(\mathrm{id} + \alpha_1) \circ \gamma_{12} \circ (\mathrm{id} + \alpha_2) \circ \gamma_{21} < \mathrm{id} \tag{1.1}$$

for suitable $\mathcal{K}_\infty$-functions α_1, α_2, then the feedback system is an ISS system with respect to the external inputs.

In this paper we concentrate on the equivalent definition of ISS in terms of ISS Lyapunov functions [31]. The small gain theorem for ISS Lyapunov functions from

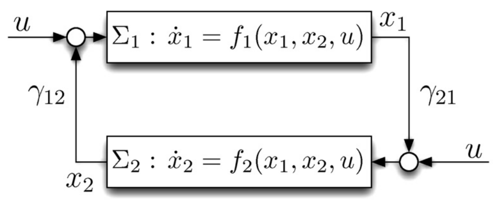

FIG. 1.1. *Feedback interconnection of two ISS systems with gains γ_{12} from Σ_2 to Σ_1 and γ_{21} from Σ_1 to Σ_2.*

[17] states that if on $(0, \infty)$ the small gain condition

$$\gamma_{12} \circ \gamma_{21} < \text{id} \tag{1.2}$$

is satisfied, then an ISS Lyapunov function may be explicitly constructed as follows. Condition (1.2) is equivalent to $\gamma_{12} < \gamma_{21}^{-1}$ on $(0, \infty)$. This permits us to construct a function $\sigma_2 \in \mathcal{K}_\infty$ such that $\gamma_{21} < \sigma_2 < \gamma_{12}^{-1}$ on $(0, \infty)$; see Figure 1.2. An ISS Lyapunov function is then defined by scaling and taking the maximum, that is, by setting $V(x) = \max\{V_1(x_1), \sigma_2^{-1}(V_2(x_2))\}$. This ISS Lyapunov function describes stability properties of the whole interconnection. In particular, given an input u, it can be seen how fast the corresponding trajectories converge to the neighborhood and how large this neighborhood is.

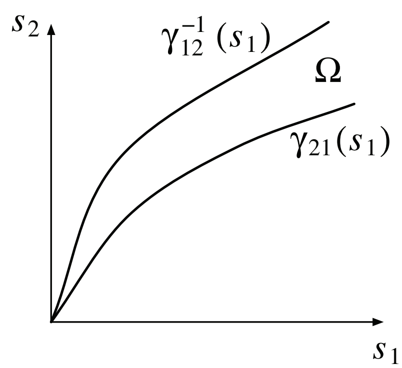

FIG. 1.2. *Two gain functions satisfying* (1.2).

At first sight the difference between the small gain conditions in (1.1) from [18] and (1.2) from [17] appears surprising. This might lead to the impression that the difference comes from studying the problem in a trajectory-based or Lyapunov-based framework. This, however, is not the case; the reason for the difference in the conditions is a result of the formulation of the ISS condition. In [18] a summation formulation was used for the trajectory-based case. In the maximization formulation of the trajectory case, the small gain condition is again (1.2) [6]. In [17] the Lyapunov formulation is investigated using maximization; the corresponding result for summation is Corollary 5.6 below, requiring condition (1.1).

In order to generalize the existing results it is useful to reinterpret the approach of [17]: note that the gains may be used to define a matrix

$$\Gamma := \begin{pmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{pmatrix},$$

which defines in a natural way a monotone operator on $\mathbb{R}^2_+$. In this way an alternative characterization of the area between γ_{21} and γ_{12}^{-1} in Figure 1.2 is that it is the area where $\Gamma(s) < s$ (with respect to the natural ordering in $\mathbb{R}^2_+$). Thus the problem of finding σ_2 may be interpreted as the problem of finding a path $\sigma : r \mapsto (r, \sigma_2(r)),\ r \in (0, \infty)$, such that $\Gamma \circ \sigma < \sigma$.

We generalize this constructive procedure for a Lyapunov function in several directions. First, the number of subsystems entering the interconnection will be arbitrary. Second, the way in which the gains of subsystem i affect subsystem j will be formulated in a general manner using the concept of *monotone aggregation functions* (MAFs). This class of functions allows for a unified treatment of summation and maximization or other ways of formulating ISS conditions. Following the matrix interpretation, this leads to a monotone operator Γ_μ on $\mathbb{R}^n_+$. The crucial thing to find is a sufficiently regular path σ such that $\Gamma_\mu \circ \sigma < \sigma$. This allows for a scaling of the Lyapunov functions for the individual subsystems to obtain one for the large scale system.

Small gain conditions on Γ_μ as in [5, 6] yield sufficient conditions that guarantee that the construction of σ can be performed. However, in [5, 6] the trajectory formulation of ISS has been studied, and the main technical ingredient was, essentially, to prove bounds on $(\mathrm{id} - \Gamma_\mu)^{-1}$. The sufficient condition for the existence of the path σ turns out to be the same, but the path itself had not been used in [5, 6]. In fact, the line of argument used there is completely different. It is shown in [24] that the results of [6] also hold for the more general ISS formulation using monotone aggregation functions. The condition requires essentially that the operator is not greater or equal to the identity in a robust sense. The construction of σ then relies on a rather delicate topological argument. What is obvious for the interconnection of two systems is not that clear in higher dimensions. It can be seen that the small gain condition imposed on the interconnection is actually a sufficient condition that allows for the application of the Knaster–Kuratowski–Mazurkiewicz theorem; see [6, 24] for further details. We show in section 9 how the construction works for three subsystems, but it is fairly clear that this methodology is not something one would like to carry out in higher dimensions. In the maximization formulation a viable alternative is the approach pursued by [20].

The construction of the Lyapunov function is explicit once the scaling function σ is known. Thus to have a really constructive procedure, a way of constructing σ is required. We do not study this problem here, but we note that based on an algorithm by Eaves [11], it is actually possible to turn this mere existence result into a (numerically) constructive method [24, 9]. Using the algorithm by Eaves and the technique of Proposition 8.8, it is then possible to construct such a vector function (but of finite length) numerically; see [24, Chapter 4]. This will be treated in more detail in future work.

The paper is organized as follows. The next section introduces the necessary notation and basic definitions, in particular, the notion of MAFs and different formulations of ISS. Section 3 gives some motivating examples that also illustrate the definitions of section 2 and explain how different MAFs occur naturally for different problems. In section 4 we introduce small gain conditions given in terms of monotone operators that naturally appear in the definition of ISS. Section 5 contains the main results, namely, the existence of the vector scaling function σ and the construction of an ISS Lyapunov function. In this section we concentrate on strongly connected networks which are easier to deal with from a technical point of view. Once this case has been resolved it is shown in section 6 how simply connected networks may be treated by studying the strongly connected components.

The actual construction of σ is given in section 8 in order to postpone the topological considerations until after applications to interconnected ISS systems have been considered in section 7. Since the topological difficulties can be avoided in the case $n = 3$, we treat this case briefly in section 9 to show a simple construction for σ. Section 10 concludes the paper.

2. Preliminaries.

2.1. Notation and conventions. Let $\mathbb{R}$ be the field of real numbers and $\mathbb{R}^n$ be the vector space of real column vectors of length n. We denote the set of nonnegative real numbers by $\mathbb{R}_+$, and $\mathbb{R}^n_+ := (\mathbb{R}_+)^n$ denotes the positive orthant in $\mathbb{R}^n$. On $\mathbb{R}^n_+$ the standard partial order is defined as follows. For vectors $v, w \in \mathbb{R}^n$ we denote

$$\begin{aligned} v \geq w &:\Longleftrightarrow\; v_i \geq w_i \text{ for } i = 1, \dots, n, \\ v > w &:\Longleftrightarrow\; v_i > w_i \text{ for } i = 1, \dots, n, \\ v \gneqq w &:\Longleftrightarrow\; v \geq w \text{ and } v \neq w. \end{aligned}$$

The maximum of two vectors or matrices is to be understood componentwise. By $|\cdot|$ we denote the 1-norm on $\mathbb{R}^n$ and by S_r, the induced sphere of radius r in $\mathbb{R}^n$ intersected with $\mathbb{R}^n_+$, which is an $(n-1)$-simplex. On $\mathbb{R}^n_+$ we denote by $\pi_I : \mathbb{R}^n_+ \to \mathbb{R}^{\#I}_+$ the *projection* of the coordinates in $\mathbb{R}^n_+$ corresponding to the indices in $I \subset \{1, \dots, n\}$ onto $\mathbb{R}^{\#I}$.

The standard scalar product in $\mathbb{R}^n$ is denoted by $\langle \cdot, \cdot \rangle$. By $U_\varepsilon(x)$ we denote the open ball of radius ε around x with respect to the Euclidean norm $\|\cdot\|$. The induced operator norm, i.e., the spectral norm, of matrices is also denoted by $\|\cdot\|$.

The space of measurable and essentially bounded functions is denoted by L^∞ with norm $\|\cdot\|_\infty$. To state the stability definitions that we are interested in, three sets of comparison functions are used: $\mathcal{K} = \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+, \gamma \text{ is continuous, strictly increasing, and } \gamma(0) = 0\}$ and $\mathcal{K}_\infty = \{\gamma \in \mathcal{K} : \gamma \text{ is unbounded}\}$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is of class $\mathcal{KL}$ if it is of class $\mathcal{K}$ in the first argument and strictly decreasing to zero in the second argument. We will call a function $V : \mathbb{R}^N \to \mathbb{R}_+$ *proper and positive definite* if there are $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that

$$\psi_1(\|x\|) \leq V(x) \leq \psi_2(\|x\|) \quad \text{for all } x \in \mathbb{R}^N .$$

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ is called *positive definite* if it is continuous and satisfies $\alpha(r) = 0$ if and only if $r = 0$.

2.2. Problem statement. We consider a finite set of interconnected systems with state $x = (x_1^T, \dots, x_n^T)^T$, where $x_i \in \mathbb{R}^{N_i}$, $i = 1, \dots, n$, and $N := \sum N_i$. For $i = 1, \dots, n$ the dynamics of the ith subsystem is given by

$$\Sigma_i : \; \dot{x}_i = f_i(x_1, \dots, x_n, u), \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}^M, \quad f_i : \mathbb{R}^{N+M} \to \mathbb{R}^{N_i}. \tag{2.1}$$

For each i we assume unique existence of solutions and forward completeness of Σ_i in the following sense. If we interpret the variables x_j, $j \neq i$, and u as unrestricted inputs, then this system is assumed to have a unique solution defined on $[0, \infty)$ for any given initial condition $x_i(0) \in \mathbb{R}^{N_i}$ and any L^∞-inputs $x_j : [0, \infty) \to \mathbb{R}^{N_j}, j \neq i$, and $u : [0, \infty) \to \mathbb{R}^M$. This can be guaranteed for instance by suitable Lipschitz and growth conditions on the f_i. It will be no restriction to assume that all systems have the same (augmented) external input u.

We write the interconnection of subsystems (2.1) as

$$\Sigma: \ \dot{x} = f(x,u), \quad f: \mathbb{R}^{N+M} \to \mathbb{R}^N. \tag{2.2}$$

Associated to such a network is a directed graph, with vertices representing the subsystems and where the directed edges (i,j) correspond to inputs going from system j to system i; see Figure 2.1. We will call the network strongly connected if its interconnection graph has the same property.

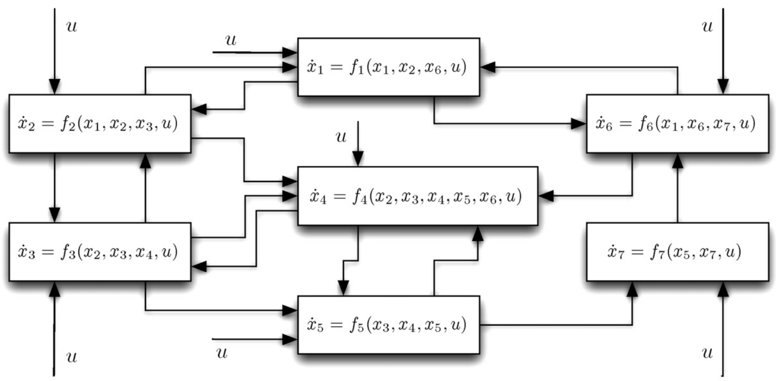

FIG. 2.1. *An example of a network of interconnected systems and the associated graph.*

For networks of the type that has just been described, we wish to construct Lyapunov functions as they are introduced now.

2.3. Stability. An appropriate stability notion to study nonlinear systems with inputs is ISS, introduced in [28]. The standard definition is as follows. A forward complete system $\dot{x} = f(x,u)$, with $x \in \mathbb{R}^N, u \in \mathbb{R}^M$, is called input-to-state stable if there are $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all initial conditions $x_0 \in \mathbb{R}^N$ and all $u \in L^\infty(\mathbb{R}_+, \mathbb{R}^M)$, we have

$$\|x(t; x_0, u(\cdot))\| \le \beta(\|x_0\|, t) + \gamma(\|u\|_\infty)\,. \tag{2.3}$$

It is known to be an equivalent requirement to ask for the existence of an ISS Lyapunov function [30]. These functions can be chosen to be smooth. For our purposes, however, it will be more convenient to have a broader class of functions available for the construction of a Lyapunov function. Thus we will call a function a *Lyapunov function candidate* if the following assumption is met.

Assumption 2.1. The function $V : \mathbb{R}^N \to \mathbb{R}_+$ is continuous, proper, and positive definite and locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$. Note that by Rademacher's theorem (e.g., [12, Theorem 5.8.6, p. 281]) locally Lipschitz continuous functions on $\mathbb{R}^N \setminus \{0\}$ are differentiable almost everywhere in $\mathbb{R}^N$.

DEFINITION 2.2. *We will call a function satisfying Assumption* 2.1 *an ISS Lyapunov function for $\dot{x} = f(x,u)$ if there exist $\gamma \in \mathcal{K}$ and a positive definite function α such that in all points of differentiability of V we have*

$$V(x) \ge \gamma(\|u\|) \quad \Longrightarrow \quad \nabla V(x) f(x,u) \le -\alpha(\|x\|)\,. \tag{2.4}$$

ISS and ISS Lyapunov functions are related in the expected manner.

THEOREM 2.3. *A system is ISS if and only if it admits an ISS Lyapunov function in the sense of Definition* 2.2.

This has been proved for smooth ISS Lyapunov functions in the literature [30]. So the hard converse statement is clear, as it is even possible to find smooth ISS Lyapunov functions which satisfy Definition 2.2. The sufficiency proof for the Lipschitz continuous case goes along the lines presented in [30, 31] using the necessary tools from nonsmooth analysis; cf. [4, Theorem 6.3].

Merely continuous ISS Lyapunov functions have been studied in [14, Chapter 3], arising as viscosity supersolutions to certain partial differential inequalities. Here we work with the Clarke generalized gradient $\partial V(x)$ of V at x. For functions V satisfying Assumption 2.1, Clarke's generalized gradient satisfies for $x \neq 0$ that

$$\partial V(x) = \operatorname{conv}\{\zeta \in \mathbb{R}^n \ : \ \text{there exists } x_k \to x : \nabla V(x_k) \text{ exists and } \nabla V(x_k) \to \zeta\}\,. \tag{2.5}$$

An equivalent formulation to (2.4) is given by

$$V(x) \geq \gamma(\|u\|) \quad \Longrightarrow \quad \text{for all } \zeta \in \partial V(x)\colon \langle \zeta, f(x,u)\rangle \leq -\alpha(\|x\|)\,. \tag{2.6}$$

Note that (2.6) is also applicable in points where V is not differentiable.

The gain γ in (2.3) is in general different from the ISS Lyapunov gain in (2.4). In the following we will always assume that gains are of class $\mathcal{K}_\infty$.

2.4. Monotone aggregation. In this paper we concentrate on the construction of ISS Lyapunov functions for the interconnected system Σ. For a single subsystem (2.1), in a similar manner to (2.4), we wish to quantify the combined effect of the inputs x_j, $j \neq i$, and u on the evolution of the state x_i. As we will see in the examples given in section 3, it depends on the system under consideration how this combined effect can be expressed: through the sum of individual effects, using the maximum of individual effects, or by other means. In order to be able to give a general treatment of this, we introduce the notion of MAFs.

DEFINITION 2.4. *A continuous function* $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ *is called a monotone aggregation function if the following three properties hold:*

(M1) *Positivity:* $\mu(s) \geq 0$ *for all* $s \in \mathbb{R}^n_+$ *and* $\mu(s) > 0$ *if* $s \gneqq 0$.
(M2) *Strict increase*[1]*: If* $x < y$*, then* $\mu(x) < \mu(y)$.
(M3) *Unboundedness: if* $\|x\| \to \infty$*, then* $\mu(x) \to \infty$.

The space of MAFs is denoted by MAF_n*, and* $\mu \in \mathrm{MAF}_n^m$ *denotes a vector* MAF*; i.e.,* $\mu_i \in \mathrm{MAF}_n$ *for* $i = 1, \dots, m$.

A direct consequence of (M2) and continuity is the following weaker monotonicity property:

(M2′) *Monotonicity:* $x \leq y \implies \mu(x) \leq \mu(y)$.

In [24, 25] MAFs have additionally been required to satisfy another property:

(M4) *Subadditivity:* $\mu(x + y) \leq \mu(x) + \mu(y)$,

which we do not need for the constructions provided in this paper, since we take a different approach; see section 6.

Standard examples of MAFs satisfying (M1)–(M4) are

$$\mu(s) = \sum_{i=1}^n s_i^l, \text{ where } l \geq 1, \qquad \text{or} \qquad \mu(s) = \max_{i=1,\dots,n} s_i \qquad \text{or}$$
$$\mu(s_1, s_2, s_3, s_4) = \max\{s_1, s_2\} + \max\{s_3, s_4\}\,.$$

[1] Compare assumption (2.10), where for the purposes of this paper, (M2) is further restricted.

On the other hand, the following function is not a MAF, since (M1) and (M3) are not satisfied: $\nu(s) = \prod_{i=1}^{n} s_i$.

Using this definition, we can define a notion of an ISS Lyapunov function for systems with multiple inputs. In this case Σ_i in (2.1) will have several gains γ_{ij} corresponding to the inputs x_j. For notational simplicity, we will include the gain $\gamma_{ii} \equiv 0$ throughout this paper. The following definition requires only Lipschitz continuity of the Lyapunov function.

DEFINITION 2.5. *Consider the interconnected system* (2.2)*, and assume that for each subsystem* Σ_j*, there is a given function* $V_j : \mathbb{R}^{N_j} \to \mathbb{R}_+$ *satisfying Assumption* 2.1*.*

For $i = 1, \dots, n$ *the function* $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ *is called an ISS Lyapunov function for* Σ_i *if there exist* $\mu_i \in \mathrm{MAF}_{n+1}, \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\}, j \neq i, \gamma_{iu} \in \mathcal{K} \cup \{0\}$*, and a positive definite function* α_i *such that at all points of differentiability of* V_i

$$V_i(x_i) \geq \mu_i\left(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u\|)\right) \quad \Longrightarrow \quad \nabla V_i(x_i) f_i(x, u) \leq -\alpha_i(\|x_i\|)\,. \tag{2.7}$$

The functions γ_{ij} *and* γ_{iu} *are called ISS Lyapunov gains.*

Several examples of ISS Lyapunov functions are given in the next section.

Let us call x_j, $j \neq i$, the *internal inputs* to Σ_i and u the *external input.* Note that the role of functions γ_{ij} and γ_{iu} is essentially to indicate whether there is any influence of different inputs on the corresponding state. In case f_i does not depend on x_j, there is no influence of x_j on the state of Σ_i. In this case we define $\gamma_{ij} \equiv 0$, in particular, always $\gamma_{ii} \equiv 0$. This allows us to collect the internal gains into a matrix

$$\Gamma := (\gamma_{ij})_{i,j=1,\dots,n}\,. \tag{2.8}$$

If we add the external gains as the last column into this matrix, then we denote it by $\overline{\Gamma}$. The function μ_i describes how the internal and external gains interactively enter in a common influence on x_i. The above definition motivates the introduction of the following nonlinear map:

$$\overline{\Gamma}_\mu : \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n, \qquad \begin{bmatrix} s_1 \\ \vdots \\ s_n \\ r \end{bmatrix} \mapsto \begin{bmatrix} \mu_1(\gamma_{11}(s_1), \dots, \gamma_{1n}(s_n), \gamma_{1u}(r)) \\ \vdots \\ \mu_n(\gamma_{n1}(s_1), \dots, \gamma_{nn}(s_n), \gamma_{nu}(r)) \end{bmatrix}. \tag{2.9}$$

Similarly, we define $\Gamma_\mu(s) := \overline{\Gamma}_\mu(s, 0)$. The matrices Γ and $\overline{\Gamma}$ are from now on referred to as *gain matrices* and Γ_μ and $\overline{\Gamma}_\mu$ as *gain operators.*

Remark 2.6 (general assumption). Given $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ and $\mu \in \mathrm{MAF}^n$, we will from now on assume that Γ and μ are *compatible* in the following sense: For each $i = 1, \dots, n$, let I_i denote the set of indices corresponding to the nonzero entries in the ith row of Γ. Then it is understood that also the restriction of μ_i to the indices I_i satisfies (M2); i.e.,

$$\mu_i(x|_{I_i}) < \mu_i(y|_{I_i}) \quad \text{if} \quad x|_{I_i} < y|_{I_i}\,. \tag{2.10}$$

In particular we assume that the function

$$s \mapsto \mu(s_1, \dots, s_n, 0), \quad s \in \mathbb{R}_+^n,$$

for $\mu \in \mathrm{MAF}_{n+1}$ satisfies (M2). Note that (M1) and (M3) are automatically satisfied.

The examples in the next section show explicitly how the introduced functions, matrices, and operators may look like for some particular cases. Clearly, the gain operators will have to satisfy certain conditions if we want to be able to deduce that (2.2) is ISS with respect to external inputs; see section 5.

3. Examples for monotone aggregation.

In this section we show how different MAFs may appear in different applications; for further examples see [10]. We begin with a purely academic example and discuss linear systems and neural networks later in this section. Consider the system

$$\dot{x} = -x - 2x^3 + \frac{1}{2}(1 + 2x^2)u^2 + \frac{1}{2}w, \tag{3.1}$$

where $x, u, w \in \mathbb{R}$. Take $V(x) = \frac{1}{2}x^2$ as a Lyapunov function candidate. It is easy to see that if $|x| \geq u^2$ and $|x| \geq |w|$, then

$$\dot{V} \leq -x^2 - 2x^4 + \frac{1}{2}x^2(1 + 2x^2) + \frac{1}{2}x^2 = -x^4 < 0$$

if $x \neq 0$. The conditions $|x| \geq u^2$ and $|x| \geq |w|$ translate into $|x| \geq \max\{u^2, |w|\}$, and in terms of V this becomes

$$V(x) \geq \max\{u^4/2, w^2/2\} \implies \dot{V}(x) \leq -x^4.$$

This is a Lyapunov ISS estimate where the gains are aggregated using a maximum; i.e., in this case we can take $\mu(s_1, s_2) = \max\{s_1, s_2\}$ and $\gamma_u(r) = r^4/2$ and $\gamma_w(r) = r^2/2$.

Note that there is a certain arbitrariness in the choice of μ and γ_{ij}. In the example one could as well take $\gamma_u(r) = \gamma_w(r) = r$ and $\mu(s_1, s_2) = \max\{s_1^4/2, s_2^2/2\}$, giving exactly the same condition, but with different gains and a different MAF. At the end of the day the small gain condition comes down to mapping properties of Γ_μ. Different choices of Γ and μ may lead to the same operator Γ_μ. However, as we will see at a later stage, certain choices of μ can be computationally more convenient than others. In particular, if we can choose $\mu = \max$, the task of checking the small gain condition reduces to checking a cycle condition; cf. section 8.4.

3.1. Linear systems.

Consider linear interconnected systems

$$\Sigma_i : \ \dot{x}_i = A_i x_i + \sum_{j=1}^{n} \Delta_{ij} x_j + B_i u_i, \quad i = 1, \ldots, n, \tag{3.2}$$

with $x_i \in \mathbb{R}^{N_i}, u_i \in \mathbb{R}^{M_i}$, and matrices A_i, B_i, Δ_{ij} of appropriate dimensions. Each system Σ_i is ISS from $(x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_n^T, u_i^T)^T$ to x_i if and only if A_i is Hurwitz. It is known that A_i is Hurwitz if and only if for any given symmetric positive definite Q_i, there is a unique symmetric positive definite solution P_i of $A_i^T P_i + P_i A_i = -Q_i$; see, e.g., [15, Corollary 3.3.47 and Remark 3.3.48, p. 284f]. Thus we choose the Lyapunov function $V_i(x_i) = x_i^T P_i x_i$, where P_i is the solution corresponding to a symmetric positive definite Q_i. In this case, along trajectories of the autonomous system

$$\dot{x}_i = A_i x_i,$$

we have

$$\dot{V}_i = x_i^T P_i A_i x_i + x_i^T A_i^T P_i x_i = -x_i^T Q_i x_i \leq -c_i \|x_i\|^2$$

for $c_i := \lambda_{\min}(Q_i) > 0$, the smallest eigenvalue of Q_i. For system (3.2) we obtain

$$\dot{V}_i = 2x_i^T P_i \left(A_i x_i + \sum_{j\neq i} \Delta_{ij} x_j + B_i u_i \right)$$

$$\leq -c_i \|x_i\|^2 + 2\|x_i\|\|P_i\| \left(\sum_{j\neq i} \|\Delta_{ij}\|\|x_j\| + \|B_i\|\|u_i\| \right) \leq -\varepsilon c_i \|x_i\|^2, \tag{3.3}$$

where the last inequality (3.3) is satisfied for a given $0 < \varepsilon < 1$ if

$$\|x_i\| \geq \frac{2\|P_i\|}{c_i(1-\varepsilon)} \left(\sum_{j\neq i} \|\Delta_{ij}\|\|x_j\| + \|B_i\|\|u\| \right), \tag{3.4}$$

with $u := (u_1^T, \dots, u_n^T)^T$. To write this implication in the form (2.7), we note that $\lambda_{\min}(P_i)\|x_i\|^2 \leq V_i(x_i) \leq \lambda_{\max}(P_i)\|x_i\|^2$. Let us denote $a_i^2 = \lambda_{\min}(P_i)$ and $b_i^2 = \lambda_{\max}(P_i) = \|P_i\|$; then the inequality (3.4) is satisfied if

$$\|P_i\| \cdot \|x_i\|^2 \geq V_i(x_i) \geq \|P_i\|^3 \left(\frac{2}{c_i(1-\varepsilon)} \right)^2 \left(\sum_{j\neq i} \frac{\|\Delta_{ij}\|}{a_j} \sqrt{V_j(x_j)} + \|B_i\|\|u\| \right)^2 .$$

This way we see that the function V_i is an ISS Lyapunov function for Σ_i with gains given by

$$\gamma_{ij}(s) = \left(\frac{2b_i^3}{c_i(1-\varepsilon)} \frac{\|\Delta_{ij}\|}{a_j} \right) \sqrt{s}$$

for $i = 1, \dots, n$, $i \neq j$, and

$$\gamma_{iu}(s) = \frac{2\|B_i\| b_i^3}{c_i(1-\varepsilon)}\, s$$

for $i = 1, \dots, n$, and $s \geq 0$. Further we have

$$\mu_i(s, r) = \left(\sum_{j=1}^n s_j + r \right)^2$$

for $s \in \mathbb{R}_+^n$ and $r \in \mathbb{R}_+$. This μ_i satisfies (M1), (M2), and (M3), but not (M4). By defining $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$, we can write

$$\overline{\Gamma} = \begin{pmatrix} 0 & \gamma_{12} & \cdots & \gamma_{1n} & \gamma_{1u} \\ \gamma_{21} & \ddots & \cdots & \gamma_{2n} & \gamma_{2u} \\ \vdots & & \ddots & \vdots & \vdots \\ \gamma_{n1} & \cdots & \gamma_{n,n-1} & 0 & \gamma_{nu} \end{pmatrix}$$

and have

$$\overline{\Gamma}_\mu(s, r) = \begin{pmatrix} \left(\frac{2b_1^3}{c_1(1-\varepsilon)} \right)^2 \left(\sum_{j\neq 1} \frac{\|\Delta_{1j}\|}{a_j} \sqrt{s_j} + \|B_1\| r \right)^2 \\ \vdots \\ \left(\frac{2b_n^3}{c_n(1-\varepsilon)} \right)^2 \left(\sum_{j\neq n} \frac{\|\Delta_{nj}\|}{a_j} \sqrt{s_j} + \|B_n\| r \right)^2 \end{pmatrix}. \tag{3.5}$$

Interestingly, the choice of quadratic Lyapunov functions for the subsystems naturally leads to a nonlinear mapping $\overline{\Gamma}_\mu$ with a useful homogeneity property; see Proposition 7.1.

3.2. Neural networks. As the next example consider a Cohen–Grossberg neural network as in [35]. The dynamics of each neuron is given by

$$\mathrm{NN}_i: \ \dot{x}_i(t) = -a_i(x_i(t)) \left(b_i(x_i(t)) - \sum_{j=1}^{n} t_{ij} s_j(x_j(t)) + J_i \right), \tag{3.6}$$

$i = 1, \dots, n, \ n \geq 2$, where x_i denotes the state of the ith neuron and a_i is a strictly positive amplification function. As in [35] we assume that the fixed point is shifted to the origin. Then the function b_i typically satisfies the sign condition $b_i(x_i)x_i \geq 0$ and satisfies furthermore $|b_i(x_i)| > \tilde{b}_i(|x_i|)$ for some $\tilde{b}_i \in \mathcal{K}_\infty$. The activation function s_i is typically assumed to be sigmoid. The matrix $T = (t_{ij})_{i,j=1,\dots,n}$ describes the interconnection of neurons in the network, and J_i is a given constant input from outside. However, for our consideration we allow J_i to be an arbitrary measurable function in L_∞.

In applications the matrix T is usually the result of training using some learning algorithm and appropriate training data. The specifics depend on the type of network architecture and learning algorithm chosen and on the particular application. Such considerations are beyond the scope of the current paper. We simply assume that T is given and concern ourselves solely with stability considerations.

Note that for any sigmoid function s_i there exists a $\gamma_i \in \mathcal{K}$ such that $|s_i(x_i)| < \gamma_i(|x_i|)$. Following [35] we assume $0 < \underline{\alpha}_i < a_i(x_i) < \overline{\alpha}_i$, $\underline{\alpha}_i, \overline{\alpha}_i \in \mathbb{R}$.

Recall the triangle inequality for $\mathcal{K}_\infty$-functions: for any $\gamma, \rho \in \mathcal{K}_\infty$ and any $a, b \geq 0$, it holds that

$$\gamma(a+b) \leq \gamma \circ (\mathrm{id} + \rho)(a) + \gamma \circ (\mathrm{id} + \rho^{-1})(b).$$

We claim that $V_i(x_i) := |x_i|$ is an ISS Lyapunov function for NN_i in (3.6). Fix an arbitrary function $\rho \in \mathcal{K}_\infty$ and some ε satisfying $\underline{\alpha}_i > \varepsilon > 0$. Then by the triangle inequality we have

$$\begin{aligned} |x_i| > \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho) \left(\frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} \sum_{j=1}^{n} |t_{ij}| \gamma_j(|x_j|) \right) + \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho^{-1}) \left(\frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} |J_i| \right) \\ \tilde{b}_i^{-1} \left(\frac{\overline{\alpha}_i}{\underline{\alpha}_i - \varepsilon} \left(\sum_{j=1}^{n} |t_{ij}| \gamma_j(|x_j|) + |J_i| \right) \right) \\ \implies \dot{V}_i = -a_i(x_i) \left(|b_i(x_i)| - \operatorname{sign} x_i \sum_{j=1}^{n} t_{ij} s_j(x_j) + \operatorname{sign} x_i J_i \right) < -\varepsilon |b_i(x_i)| . \end{aligned}$$

In this case we have

$$\mu_i(s, r) = \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho)(s_1 + \cdots + s_n) + \tilde{b}_i^{-1} \circ (\mathrm{id} + \rho^{-1})(r)$$

which is additive with respect to the external input and

$$\gamma_{ij} = \frac{\overline{\alpha}_i |t_{ij}|}{\underline{\alpha}_i - \varepsilon} \gamma_j(|x_j|), \quad \gamma_{iu} = \frac{\overline{\alpha}_i \mathrm{id}}{\underline{\alpha}_i - \varepsilon}.$$

The MAF μ_i satisfies (M1), (M2), and (M3). It satisfies (M4) if and only if $(\tilde{b}_i)^{-1}$ is subadditive.

4. Monotone operators and generalized small gain conditions. In section 2.4 we saw that in the ISS context the mutual influence between subsystems (2.1) and the influence from external inputs to the subsystems can be quantified by the gain matrices Γ and $\overline{\Gamma}$ and gain operators Γ_μ and $\overline{\Gamma}_\mu$. The interconnection structure of the subsystems naturally leads to a weighted, directed graph, where the weights are the nonlinear gain functions and the vertices are the subsystems. There is an edge from the vertex i to the vertex j if and only if there is an influence of the state x_i on the state x_j; i.e., there is a nonzero gain γ_{ji}.

Connectedness properties of the interconnection graph together with mapping properties of the gain operators will yield a generalized small gain condition. In essence we need a nonlinear version of a Perron vector for the construction of a Lyapunov function for the interconnected system. This will be made rigorous in the following. But first we introduce some further notation.

The adjacency matrix $A_\Gamma = (a_{ij})$ of a matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ is defined by $a_{ij} = 0$ if $\gamma_{ij} \equiv 0$, and $a_{ij} = 1$, otherwise. Then $A_\Gamma = (a_{ij})$ is also the adjacency matrix of the graph representing an interconnection.

We say that a matrix Γ is *primitive, irreducible*, or *reducible* if and only if A_Γ is primitive, irreducible, or reducible, respectively. Recall (and see [2] for more on this subject) that a nonnegative matrix A is

- *primitive* if there exists a $k \geq 1$ such that A^k is positive;
- *irreducible* if for every pair (i,j), there exists a $k \geq 1$ such that the (i,j)th entry of A^k is positive; obviously, primitivity implies irreducibility;
- *reducible* if it is not irreducible.

A network or a graph is strongly connected if and only if the associated adjacency matrix is irreducible; see also [2].

For $\mathcal{K}_\infty$-functions $\alpha_1, \dots, \alpha_n$ we define a diagonal operator $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$D(s) := (s_1 + \alpha_1(s_1), \dots, s_n + \alpha_n(s_n))^T, \quad s \in \mathbb{R}^n_+. \tag{4.1}$$

For an operator $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, the condition $T \not\geq \mathrm{id}$ means that for all $s \neq 0$, $T(s) \not\geq s$. In words, at least one component of $T(s)$ has to be strictly less than the corresponding component of s.

Definition 4.1 (small gain conditions). *Let a gain matrix* Γ *and a monotone aggregation* μ *be given. The operator* Γ_μ *is said to satisfy the small gain condition* (SGC) *if*

$$\Gamma_\mu \not\geq \mathrm{id}. \tag{SGC}$$

Furthermore, Γ_μ *satisfies the strong small gain condition* (sSGC) *if there exists a* D *as in* (4.1) *such that*

$$D \circ \Gamma_\mu \not\geq \mathrm{id}. \tag{sSGC}$$

It is not difficult to see that (sSGC) can equivalently be stated as

$$\Gamma_\mu \circ D \not\geq \mathrm{id}. \tag{sSGC$'$}$$

Also for (sSGC) or (sSGC′) to hold it is sufficient to assume that the function $\alpha_1, \dots, \alpha_n$ are all identical. This can be seen by defining $\alpha(s) := \min_i \alpha_i(s)$. We abbreviate this by writing $D = \mathrm{diag}(\mathrm{id} + \alpha)$ for some $\alpha \in \mathcal{K}_\infty$.

For maps $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ we define the following sets:

$$\Omega(T) := \{s \in \mathbb{R}^n_+ : T(s) < s\} = \bigcap_{i=1}^{n} \Omega_i(T), \text{ where}$$
$$\Omega_i(T) := \{s \in \mathbb{R}^n_+ : T(s)_i < s_i\}.$$

If no confusion arises we will omit the reference to T. Topological properties of the introduced sets are related to (SGC), (sSGC), and (sSGC′); cf. also [5, 6, 25]. They will be used in the next section for the construction of an ISS Lyapunov function for the interconnection.

5. Lyapunov functions. In this section we present the two main results of the paper. The first is a topological result on the existence of a jointly unbounded path in the set Ω, provided that Γ_μ satisfies the small gain condition. This path will be crucial in the construction of a Lyapunov function, which is the second main result of this section.

DEFINITION 5.1. *A continuous path $\sigma \in \mathcal{K}^n_\infty$ will be called an Ω-path with respect to Γ_μ if*

(i) *for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;*

(ii) *for every compact set $K \subset (0, \infty)$, there are constants $0 < c < C$ such that for all $i = 1, \ldots, n$ and all points of differentiability of σ_i^{-1}, we have*

$$0 < c \le (\sigma_i^{-1})'(r) \le C \quad \text{for all } r \in K; \tag{5.1}$$

(iii) *$\sigma(r) \in \Omega(\Gamma_\mu)$ for all $r > 0$; i.e.,*

$$\Gamma_\mu(\sigma(r)) < \sigma(r) \quad \text{for all } r > 0. \tag{5.2}$$

Now we can state the first of our two main results, which regards the existence of Ω-paths.

THEOREM 5.2. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be a gain matrix and $\mu \in \mathrm{MAF}^n_n$. Assume that one of the following assumptions is satisfied:*

(i) *Γ_μ is linear and the spectral radius of Γ_μ is less than one.*

(ii) *Γ is irreducible and $\Gamma_\mu \not\ge \mathrm{id}$.*

(iii) *$\mu = \max$ and $\Gamma_\mu \not\ge \mathrm{id}$.*

(iv) *Alternatively, assume that Γ_μ is bounded (i.e., $\Gamma \in ((\mathcal{K} \setminus \mathcal{K}_\infty) \cup \{0\})^{n\times n}$) and satisfies $\Gamma_\mu \not\ge \mathrm{id}$.*

Then there exists an Ω-path σ with respect to Γ_μ.

We will postpone the proof of this rather topological result to section 8 and reap the fruits of Theorem 5.2 first. Note, however, that for Theorem 5.2 there exists a "cycle gain < id"-type equivalent formulation; cf. Theorem 8.14 and see [21, 33, 6, 20].

In addition to the above result, the existence of Ω-paths can also be asserted for reducible Γ and Γ with mixed, bounded and unbounded, class $\mathcal{K}$ entries; see Theorem 8.12 and Proposition 8.13, respectively.

THEOREM 5.3. *Consider the interconnected system Σ given by* (2.1) *and* (2.2) *where each of the subsystems Σ_i has an ISS Lyapunov function V_i, the corresponding gain matrix is given by* (2.8)*, and $\mu = (\mu_1, \ldots, \mu_n)^T$ is given by* (2.7)*. Assume there are an Ω-path σ with respect to Γ_μ and a function $\varphi \in \mathcal{K}_\infty$ such that*

$$\overline{\Gamma}_\mu(\sigma(r), \varphi(r)) < \sigma(r) \quad \text{for all } r > 0 \tag{5.3}$$

is satisfied, then an ISS Lyapunov function for the overall system is given by

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)) . \tag{5.4}$$

In particular, for all points of differentiability of V we have the implication

$$V(x) \geq \max\{\varphi^{-1}(\gamma_{iu}(\|u\|)) \mid i = 1,\dots,n\} \implies \nabla V(x) f(x,u) \leq -\alpha(\|x\|), \tag{5.5}$$

where α is a suitable positive definite function.

Note that by construction the Lyapunov function V is not smooth, even if the functions V_i for the subsystems are. This is why it is appropriate in this framework to consider Lipschitz continuous Lyapunov functions, which are differentiable almost everywhere.

Proof. We will show the assertion in the Clarke gradient sense. For $x = 0$ there is nothing to show. So let $0 \neq x = (x_1^T, \dots, x_n^T)^T$. Denote by I the set of indices i for which

$$V(x) = \sigma_i^{-1}(V_i(x_i)) \geq \max_{j \neq i} \sigma_j^{-1}(V_j(x_j)) . \tag{5.6}$$

Then $x_i \neq 0$ for $i \in I$. Also as V is obtained through maximization, we have because of [4, p. 83] that

$$\partial V(x) \subset \operatorname{conv} \left\{ \bigcup_{i \in I} \partial[\sigma_i^{-1} \circ V_i \circ \pi_i](x) \right\} . \tag{5.7}$$

Fix $i \in I$, and assume without loss of generality $i = 1$. Then if we assume $V(x) \geq \max_{i=1,\dots,n}\{\varphi^{-1}(\gamma_{iu}(\|u\|))\}$, it follows in particular that $\gamma_{1u}(\|u\|) \leq \varphi(V(x))$. Using the abbreviation $r := V(x)$, denoting the first component of $\overline{\Gamma}_\mu$ by $\overline{\Gamma}_{\mu,1}$, and using assumption (5.3), we have

$$\begin{aligned} V_1(x_1) = \sigma_1(r) &> \overline{\Gamma}_{\mu,1}(\sigma(r), \varphi(r)) \\ &= \mu_1 \left[\gamma_{11}(\sigma_1(r)), \dots, \gamma_{1n}(\sigma_n(r)), \varphi(r)\right] \\ &\geq \mu_1 \left[\gamma_{11}(\sigma_1(r)), \dots, \gamma_{1n}(\sigma_n(r)), \gamma_{1u}(\|u\|)\right] \\ &= \mu_1 \left[\gamma_{11} \circ \sigma_1 \circ \sigma_1^{-1}(V_1(x_1)), \dots, \gamma_{1n} \circ \sigma_n \circ \sigma_1^{-1}(V_1(x_1)), \gamma_{1u}(\|u\|)\right] \\ &\geq \mu_1 \left[\gamma_{11} \circ V_1(x_1), \dots, \gamma_{1n} \circ V_n(x_n), \gamma_{1u}(\|u\|)\right] , \end{aligned}$$

where we have used (5.6) and (M2$'$) in the last inequality. Thus the ISS condition (2.7) is applicable, and we have for all $\zeta \in \partial V_1(x_1)$ that

$$\langle \zeta, f_1(x,u) \rangle \leq -\alpha_1(\|x_1\|). \tag{5.8}$$

By the chain rule for Lipschitz continuous functions [4, Theorem 2.5], we have

$$\partial(\sigma_i^{-1} \circ V_i)(x_i) \subset \{c\zeta \ : \ c \in \partial\sigma_i^{-1}(y) \, , \ y = V_i(x_i) \, , \ \zeta \in \partial V_i(x_i)\} .$$

Note that in the previous equation the number c is bounded away from zero because of (5.1). We set for $\rho > 0$

$$\tilde{\alpha}_i(\rho) := c_{\rho,i} \, \alpha_i(\rho) > 0 \, ,$$

where $c_{\rho,i}$ is the constant corresponding to the set $K := \{x_i \in \mathbb{R}^{N_i} \;:\; \rho/2 \leq \|x_i\| \leq 2\rho\}$ given by (5.1) in the definition of an Ω-path. With the convention $x = (x_1^T, \dots, x_n^T)^T$ we now define for $r > 0$

$$\alpha(r) = \min\{\tilde{\alpha}_i(\|x_i\|) \mid \|x\| = r, V(x) = \sigma_i^{-1}(V_i(x_i))\} > 0\,.$$

Here we have used that for a given $r > 0$ and $\|x\| = r$, the norm of $\|x_i\|$ such that $V(x) = \sigma_i^{-1}(V_i(x_i))$ is bounded away from 0.

It now follows from (5.8) that if $V(x) \geq \max_{i=1,\dots,n}\{\varphi^{-1}(\gamma_{iu}(\|u\|))\}$, then we have for all $\zeta \in \partial\left[\sigma_1^{-1} \circ V_1\right](x_1)$ that

$$\langle \zeta, f_1(x,u)\rangle \leq -\alpha(\|x\|)\,. \tag{5.9}$$

In particular, the right-hand side depends on x and not only on x_1. The same argument applies for all $i \in I$. Now for any $\zeta \in \partial V(x)$ we have by (5.7) that $\zeta = \sum_{i\in I} \lambda_i c_i \zeta_i$ for suitable $\lambda_i \geq 0, \sum_{i\in I}\lambda_i = 1$, and with $\zeta_i \in \partial(V_i \circ \pi_i)(x)$ and $c_i \in \partial\sigma_i^{-1}(V_i(x_i))$. It follows that

$$\begin{aligned}\langle \zeta, f(x,u)\rangle = \sum_{i\in I}\lambda_i\langle c_i\zeta_i, f(x,u)\rangle &= \sum_{i\in I}\lambda_i\langle c_i\pi_i(\zeta_i), f_i(x,u)\rangle \\ &\leq -\sum_{i\in I}\lambda_i\alpha(\|x\|) = -\alpha(\|x\|)\,.\end{aligned}$$

This shows the assertion. ∎

In the absence of external inputs, ISS is the same as 0-GAS (cf. [29, 30, 31]). We note the following consequence in the case that only global asymptotic stability is of interest.

COROLLARY 5.4 (0-GAS for strongly interconnected networks). *In the setting of Theorem* 5.3*, assume that the external inputs satisfy* $u \equiv 0$ *and that the network of interconnected systems is strongly connected. If* $\Gamma_\mu \not\geq \mathrm{id}$*, then the network is 0-GAS.*

Proof. By Theorem 5.2(ii) there exists an Ω-path, and a nonsmooth Lyapunov for the network is given by (5.4); hence, the origin of the externally unforced composite system is globally asymptotically stable. ∎

Remark 5.5. At first sight it might seem that the previous corollary is stronger than [18, Corollary 2.1], as no robustness term D is needed in the assumptions. However, the result here is formulated for Lyapunov functions, whereas the result in [18] is based on the trajectory formulation of ISS in summation form. The proof in the trajectory version essentially requires bounds on $(\mathrm{id} - \Gamma_\mu)^{-1}$, which relies heavily on D unless $\mu = \max$ [18, 6, 24]. In contrast, for 0-GAS the D is not needed in the Lyapunov setting, because for irreducible Γ it is possible to construct the path σ without D by Theorem 5.2(ii).

We now specialize Theorem 5.3 to particular cases of interest. Namely, when the gain with respect to the external input u enters the ISS condition (i) additively, (ii) via maximization, and (iii) as a factor.

COROLLARY 5.6 (additive gain of external input $\mathbf{u}$). *Consider the interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i *and the corresponding gain matrix is given by* (2.9)*. Assume that the ISS condition is additive in the gain of* u*; that is,*

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \Gamma_\mu(V_1(x_1), \dots, V_n(x_n)) + \gamma_u(\|u\|)\,, \tag{5.10}$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \dots, \gamma_{nu}(\|u\|))^T$*. If* Γ_μ *is irreducible and if there exists an* $\alpha \in \mathcal{K}_\infty$ *such that for* $D = \mathrm{diag}(\mathrm{id} + \alpha)$ *the gain operator* Γ_μ *satisfies the strong small gain condition*

$$D \circ \Gamma_\mu(s) \not\geq s,$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (5.4)*, where* $\sigma \in \mathcal{K}_\infty^n$ *is an arbitrary* Ω*-path with respect to* $D \circ \Gamma_\mu$*.*

Proof. By Theorem 5.2 an $\Omega(D \circ \Gamma_\mu)$-path σ exists. Observe that by irreducibility, (M1), and (M3) it follows that $\Gamma_\mu(\sigma)$ is unbounded in all components. Let $\varphi \in \mathcal{K}_\infty$ be such that for all $r \geq 0$

$$\min_{i=1,\dots,n} \{\alpha(\Gamma_{\mu,i}(\sigma(r)))\} \geq \max_{i=1,\dots,n} \{\gamma_{iu}(\varphi(r))\} .$$

Note that this is possible because on the left we take the minimum of a finite number of $\mathcal{K}_\infty$-functions. Then we have for all $r > 0$, $i = 1, \dots, n$, that

$$\sigma_i(r) > D \circ \Gamma_{\mu,i}(\sigma(r)) = \Gamma_{\mu,i}(\sigma(r)) + \alpha(\Gamma_{\mu,i}(\sigma(r))) \geq \Gamma_{\mu,i}(\sigma(r)) + \gamma_{iu}(\varphi(r)) .$$

Thus $\sigma(r) > \overline{\Gamma}_\mu(\sigma(r), \varphi(r))$ and the assertion follows from Theorem 5.3. □

Corollary 5.7 (maximization with respect to external gain). *Consider the interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i *and the corresponding gain matrix is given by* (2.9)*. Assume that* u *enters the ISS condition via maximization; that is,*

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = \max\left\{\Gamma_\mu(V_1(x_1), \dots, V_n(x_n)), \gamma_u(\|u\|)\right\} , \tag{5.11}$$

where $\gamma_u(\|u\|) = (\gamma_{1u}(\|u\|), \dots, \gamma_{nu}(\|u\|))^T$*. Then, if* Γ_μ *is irreducible and satisfies the small gain condition*

$$\Gamma_\mu(s) \not\geq s,$$

the interconnected system is ISS and an ISS Lyapunov function is given by (5.4)*, where* $\sigma \in \mathcal{K}_\infty^n$ *is an arbitrary* Ω*-path with respect to* Γ_μ *and* φ *is a* $\mathcal{K}_\infty$*-function with the property*

$$\gamma_{iu} \circ \varphi(r) \leq \Gamma_{\mu,i}(\sigma(r)) , \quad i = 1, \dots, n, \tag{5.12}$$

where $\Gamma_{\mu,i}$ *denotes the ith row of* Γ_μ*.*

Proof. By Theorem 5.2 an $\Omega(\Gamma_\mu)$-path σ exists. Note that by irreducibility, (M1), and (M3) it follows that $\Gamma_\mu(\sigma)$ is unbounded in all components. Hence $\varphi \in \mathcal{K}_\infty$ satisfying (5.12) exists, and we obtain

$$\sigma(r) > \max\{\, \Gamma_\mu(\sigma(r)), \gamma_u(\varphi(r))\} .$$

This is (5.3) for the case of maximization of gains in u. The claim follows from Theorem 5.3. □

In the next result observe that (M3) is not always necessary for the u-component of μ.

Corollary 5.8 (separation in gains). *Consider the interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i *and the corresponding gain matrix* Γ *is given by* (2.9)*. Assume that* Γ *is irreducible* *and that the gains in the ISS condition are separated; that is, there exist* $\mu \in \mathrm{MAF}_n^n$, $c \in \mathbb{R},\ c > 0$, *and* $\gamma_u \in \mathcal{K}_\infty$ *such that*

$$\overline{\Gamma}_\mu(V_1(x_1), \dots, V_n(x_n), \|u\|) = (c + \gamma_u(\|u\|))\ \Gamma_\mu(V_1(x_1), \dots, V_n(x_n)) . \tag{5.13}$$

If there exists an $\alpha \in \mathcal{K}_\infty$ *such that for* $D = \mathrm{diag}(c \cdot \mathrm{id} + \mathrm{id} \cdot \alpha)$ *the gain operator* Γ_μ *satisfies the strong small gain condition*

$$D \circ \Gamma_\mu(s) \not\geq s,$$

then the interconnected system is ISS and an ISS Lyapunov function is given by (5.4), *where* $\sigma \in \mathcal{K}_\infty^n$ *is an arbitrary* Ω*-path with respect to* $D \circ \Gamma_\mu(s)$.

Proof. If Γ_μ is irreducible, then also $D \circ \Gamma_\mu$ is irreducible, and so by Theorem 5.2(ii) an $\Omega(D \circ \Gamma_\mu)$-path σ exists. Let $\varphi \in \mathcal{K}_\infty$ be such that for all $r \geq 0$

$$\varphi(r) \leq \min_{i=1,\dots,n} \{\gamma_u^{-1} \circ \alpha \circ \Gamma_{\mu,i}(\sigma(r))\} ,$$

where, as in the previous corollaries, we appeal to irreducibility, (M1), and (M3). Then for each i we have

$$\sigma_i(r) > \Gamma_{\mu,i}(\sigma(r))(c + \alpha(\Gamma_{\mu,i}(\sigma(r)))) \geq \Gamma_{\mu,i}(\sigma(r))(c + \gamma_u \circ \varphi(r)),$$

and hence

$$\sigma(r) > (c + \gamma_u(\varphi(r)))\Gamma_\mu(\sigma(r)) = \overline{\Gamma}_\mu(\sigma(r), \varphi(r));$$

the assertion follows from (5.13) and Theorem 5.3. ∎

6. The reducible case and scaling. The results that have been obtained so far concern mostly strongly connected networks, that is, networks with an irreducible gain operator. Already in [27] it has been shown that cascades of ISS systems are ISS. Cascades are a special case of networks where the gain matrix is reducible. In this section we briefly explain how a Lyapunov function for a network that is not strongly connected may be constructed based on the construction for the strongly connected components of the network. Another approach would be to construct the Ω-path for reducible operators Γ_μ as has been done in [25] using assumption (M4).

It is well known that if the network is not strongly connected or, equivalently, if the gain matrix Γ is reducible, then Γ may be brought in upper block triangular form via a permutation of the vertices of the network as in the nonnegative matrix case [2, 6]. After this transformation $\overline{\Gamma}$ is of the form

$$\overline{\Gamma} = \begin{bmatrix} \Upsilon_{11} & \Upsilon_{12} & \dots & \Upsilon_{1d} & \Upsilon_{1u} \\ 0 & \Upsilon_{22} & \dots & \Upsilon_{2d} & \Upsilon_{2u} \\ \vdots & & \ddots & & \\ 0 & \dots & 0 & \Upsilon_{dd} & \Upsilon_{du} \end{bmatrix} , \tag{6.1}$$

where each of the blocks on the diagonal $\Upsilon_{jj} \in (\mathcal{K}_\infty \cup \{0\})^{d_j \times d_j}$, $j = 1, \dots, d$, is either irreducible or 0. Let $q_j = \sum_{l=1}^{j-1} d_l$, with the convention that $q_1 = 0$. We denote the states corresponding to the strongly connected components by

$$z_j^T = \begin{bmatrix} x_{q_j+1}^T, & x_{q_j+2}^T, & \dots, & x_{q_{j+1}}^T \end{bmatrix} .$$

We will show that in order to obtain an overall ISS Lyapunov function, it is sufficient to construct ISS Lyapunov functions for each of the irreducible blocks (where the respective states with higher indices are treated as inputs). The desired result is an iterative application of the following observation.

LEMMA 6.1. *Let a gain matrix* $\overline{\Gamma} \in (\mathcal{K}_\infty \cup \{0\})^{2\times 3}$ *be given by*

$$\overline{\Gamma} = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{1u} \\ 0 & 0 & \gamma_{2u} \end{bmatrix}, \tag{6.2}$$

and let $\overline{\Gamma}_\mu$ *be defined by* $\mu \in \mathrm{MAF}_3^2$. *Then there exist an* Ω*-path* σ *and* $\varphi \in \mathcal{K}_\infty$ *such that* (5.3) *holds.*

Proof. By construction the maps $\eta_1 : r \mapsto \mu_1(\gamma_{12}(r), \gamma_{1u}(r))$ and $\eta_2 : r \mapsto \mu_2(\gamma_{12}(u))$ are in $\mathcal{K}_\infty$. Choose a $\mathcal{K}_\infty$-function $\tilde{\eta}_1 \geq \eta_1$ such that $\tilde{\eta}_1$ satisfies the conditions (i) and (ii) in Definition 5.1. Define $\sigma(r) = \begin{bmatrix} 2\tilde{\eta}_1(r) & r \end{bmatrix}^T$ and $\varphi(r) := \min\{r, \eta_2^{-1}(r/2)\}$. Then it is a straightforward calculation to check that the assertion holds. ∎

The result is now as follows.

PROPOSITION 6.2. *Consider a simply connected interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i*, the corresponding gain matrix is given by* (2.8)*, and* $\mu = (\mu_1, \ldots, \mu_n)^T$ *is given by* (2.7). *Assume that the gain matrix* $\overline{\Gamma}$ *is in the reduced form* (6.1). *If for each* $j = 1, \ldots, d-1$ *there exists an ISS Lyapunov function* W_j *for the state* z_j *with respect to the inputs* $z_{j+1}, \ldots, z_d, u$*, then there exists an ISS Lyapunov function* V *for the state* x *with respect to the input* u.

Proof. By assumption for each $j = 1, \ldots, d-1$ there exist gain functions $\chi_{jk} \in \mathcal{K}_\infty$ and $\chi_{ju} \in \mathcal{K}_\infty$ and MAFs $\tilde{\mu}_j$ such that

$$\begin{aligned} W_j(z_j) \geq \tilde{\mu}_j(\chi_{jj+1}(W_{j+1}(z_{j+1})), \ldots, \chi_{jd}(W_d(z_d)), \chi_{ju}(\|u\|)) \\ \implies \nabla W_j(z_j) f_j(z_j, z_{j+1}, \ldots, z_d, u) < -\tilde{\alpha}_j(\|z_j\|). \end{aligned}$$

We now argue by induction. If $d = 1$, there is nothing to show. If the result is shown for $d-1$ blocks, consider a gain matrix as in (6.1). By assumption there exists an ISS Lyapunov function V_{d-1} such that

$$\begin{aligned} V_{d-1}(z_{d-1}) \geq \mu_1(\gamma_{12}(V_d(z_d)), \gamma_{1u}(\|u\|)) \\ \implies \nabla V_{d-1}(z_{d-1}) f_{d-1}(z_{d-1}, z_d, u) \leq -\alpha_{d-1}(\|z_{d-1}\|). \end{aligned}$$

As the remaining part has only external inputs, we see that $\overline{\Gamma}$ is of the form (6.2), and so Lemma 6.1 is applicable. This shows that the assumptions of Theorem 5.3 are met, and so a Lyapunov function for the overall system is given by (5.4). ∎

It is easy to see that the assumption $\Gamma_\mu \not\geq \mathrm{id}$ (or $\Gamma_\mu \circ D \not\geq \mathrm{id}$) is equivalent to the requirement that the blocks Υ_{jj} on the diagonal satisfy the (strong) small gain condition (SGC) (or (sSGC)). Thus we immediately obtain the following statements.

COROLLARY 6.3 (summation of gains). *Consider the interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i *and the corresponding gain matrix is given by* (2.9). *Assume that the ISS condition is additive in the gains; that is,*

$$\overline{\Gamma}_{\mu,i}(V_1(x_1), \ldots, V_n(x_n), \|u\|) = \sum_{j=1}^{n} \gamma_{ij}(V_j(x_j)) + \gamma_{iu}(\|u\|). \tag{6.3}$$

If there exists an $\alpha \in \mathcal{K}_\infty$ *such that for* $D = \text{diag}(\text{id}+\alpha)$ *the gain operator* Γ_μ *satisfies the strong small gain condition*

$$D \circ \Gamma_\mu(s) \not\geq s,$$

then the interconnected system is ISS.

Proof. After permutation $\overline{\Gamma}$ is of the form (6.1). For each of the diagonal blocks Corollary 5.6 is applicable, and the result follows from Proposition 6.2. ∎

COROLLARY 6.4 (maximization of gains). *Consider the interconnected system* Σ *given by* (2.1) *and* (2.2) *where each of the subsystems* Σ_i *has an ISS Lyapunov function* V_i *and the corresponding gain matrix is given by* (2.9). *Assume that the gains enter the ISS condition via maximization; that is,*

$$\overline{\Gamma}_{\mu,i}(V_1(x_1), \dots, V_n(x_n), \|u\|) = \max\left\{\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u\|)\right\}. \tag{6.4}$$

If the gain operator Γ_μ *satisfies the small gain condition*

$$\Gamma_\mu(s) \not\geq s,$$

then the interconnected system is ISS.

Proof. After permutation $\overline{\Gamma}$ is of the form (6.1). For each of the diagonal blocks Corollary 5.7 is applicable, and the result follows from Proposition 6.2. ∎

7. Applications of the general small gain theorem. In section 3 we presented several examples of functions μ_i, γ_i and gain operators Γ_μ, $\overline{\Gamma}_\mu$. Here we will show how our main results apply to these examples. Before we proceed, let us consider the special case of homogeneous Γ_μ (of degree 1) [13]. Here Γ_μ is homogeneous of degree 1 if for any $s \in \mathbb{R}^n_+$ and any $r > 0$ we have $\Gamma_\mu(rs) = r\Gamma_\mu(s)$.

PROPOSITION 7.1 (explicit paths and Lyapunov functions for homogeneous gain operators). *Let* Σ *in* (1.2) *be a strongly connected network of subsystems* (1.1) *and* Γ_μ, $\overline{\Gamma}_\mu$ *be the corresponding gain operators. Let* Γ_μ *be homogeneous, and let* $\overline{\Gamma}_\mu$ *satisfy one of the conditions* (6.3), (6.4), *or* (5.13). *If* Γ_μ *satisfies the strong small gain condition* (sSGC) *(*(SGC) *in case of* (6.4)*), then the interconnection* Σ *is ISS; moreover, there exists a (nonlinear) eigenvector* $0 < s \in \mathbb{R}^n$ *of* Γ_μ *such that* $\Gamma_\mu(s) = \lambda s$ *with* $\lambda < 1$, *and an ISS Lyapunov function for the network is given by*

$$V(x) = \max_i\{V_i(x_i)/s_i\}. \tag{7.1}$$

Proof. First note that either Corollary 6.3, 6.4, or 5.8 can be applied, and the ISS property follows immediately. By the assumptions of the proposition we have an irreducible monotone homogeneous operator Γ_μ on the positive orthant $\mathbb{R}^n_+$. By the generalized Perron–Frobenius theorem [13] there exists a positive eigenvector $s \in \mathbb{R}^n_+$. Its eigenvalue λ is less than one; otherwise, we have a contradiction to the small gain condition. The ray defined by this vector s is a corresponding Ω-path and by Theorem 5.3 we obtain (7.1). ∎

One type of homogeneous operator arises from linear operators through multiplicative coordinate transforms. In this case we can further specialize the assumptions of the previous result.

LEMMA 7.2. *Let* $\alpha \in \mathcal{K}_\infty$ *satisfy*[2] $\alpha(ab) = \alpha(a)\alpha(b)$ *for all* $a, b \geq 0$. *Let* $D = \text{diag}(\alpha)$, $G \in \mathbb{R}^{n\times n}_+$, *and* Γ_μ *be given by*

$$\Gamma_\mu(s) = D^{-1}(GD(s)).$$

[2] In other words, $\alpha(r) = r^c$ for some $c > 0$.

Then Γ_μ is homogeneous. Moreover, $\Gamma_\mu \not\geq \mathrm{id}$ if and only if the spectral radius of G is less than one.

Proof. If the spectral radius of G is less than one, then there exists a positive vector $\tilde{s}$ satisfying $G\tilde{s} < \tilde{s}$: just add a small $\delta > 0$ to every entry of G so that the spectral radius $\rho(\tilde{G})$ of $\tilde{G}$ is still less than one, due to continuity of the spectrum. Then there exists a Perron vector $\tilde{s}$ such that $G\tilde{s} < \tilde{G}\tilde{s} = \rho(\tilde{G})\tilde{s} < \tilde{s}$. Define $\hat{s} = D^{-1}(\tilde{s}) > 0$, and observe that $\alpha^{-1}(ab) = \alpha^{-1}(a)\alpha^{-1}(b)$. Then we have

$$\Gamma_\mu(r\hat{s}) = D^{-1}(GD(r\hat{s})) = D^{-1}(\alpha(r)GD(\hat{s})) = rD^{-1}(G\tilde{s}) < rD^{-1}(\tilde{s}) = r\hat{s} \tag{7.2}$$

for all $r > 0$. So an Ω-path for Γ_μ is given by $\sigma(r) = r\hat{s}$ for $r \geq 0$. Existence of an Ω-path implies the small gain condition: the origin in $\mathbb{R}^n_+$ is globally attractive with respect to the system $s^{k+1} = \Gamma_\mu(s^k)$, as can be seen by a monotonicity argument. By [6, Theorem 23] or [25, Proposition 4.1] we have $\Gamma_\mu \not\geq \mathrm{id}$.

Assuming that the spectral radius of G is greater or equal to one, there exists $\tilde{s} \in \mathbb{R}^n_+$, $\tilde{s} \neq 0$, such that $G\tilde{s} \geq \tilde{s}$. Defining $\hat{s} = D^{-1}(\tilde{s})$, we have $\Gamma_\mu(\hat{s}) = D^{-1}(GD(\hat{s})) = D^{-1}(G\tilde{s}) \geq D^{-1}(\tilde{s}) = \hat{s}$. Hence $\Gamma_\mu \not\geq \mathrm{id}$ if and only if the spectral radius of G is less than one.

Homogeneity of Γ_μ is obtained as in (7.2). ∎

7.1. Application to linear interconnected systems. Consider the interconnection (3.2) of linear systems from section 3.1.

PROPOSITION 7.3. *Let each Σ_i in (3.2) be ISS with a quadratic ISS Lyapunov function V_i so that the corresponding operator Γ_μ can be taken to be as in (3.5). If the spectral radius $r(G)$ of the associated matrix*

$$G = \left(\frac{2b_i^3 \|\Delta_{ij}\|}{c_i(1-\varepsilon)a_j} \right)_{ij} \tag{7.3}$$

is less than one, then the interconnection

$$\Sigma: \quad \dot{x} = (A + \Delta)x + Bu$$

is ISS and its (nonsmooth) ISS Lyapunov function can be taken as

$$V(x) = \max_i \frac{1}{s_i} x_i^T P_i x_i$$

for some positive vector $s \in \mathbb{R}^n_+$.

Proof. We have $\Gamma_\mu = D^{-1}(GD(\cdot))$, where $D = \mathrm{diag}(\alpha)$ for $\alpha(r) = \sqrt{r}$. Now α satisfies the assumptions of Lemma 7.2, which yields that Γ_μ satisfies the small gain condition $\Gamma_\mu \not\geq \mathrm{id}$ if and only if $r(G) < 1$. If G or, equivalently, Γ_μ is irreducible, then there exists by Proposition 7.1 an $s > 0$ such that $\Gamma_\mu(s) < s$. By (3.5) we see that there exists an $r^* \in (0, \infty)$ such that $\overline{\Gamma}_\mu(s, r^*) < s$. Then defining $\sigma(r) = rs$ and $\varphi(r) = \sqrt{r}r^*$ we obtain for all $r > 0$ that

$$\overline{\Gamma}_\mu(\sigma(r), \varphi(r)) = r\overline{\Gamma}_\mu(s, r^*) < rs = \sigma(r)\,.$$

Thus the conditions of Theorem 5.3 are satisfied, and an ISS Lyapunov function can be taken as $V(x) = \max_i \frac{1}{s_i} x_i^T P_i x_i$.

If G is reducible, the previous construction has to be performed for every irreducible block and then the scaling techniques of section 6 need to be applied. ∎

7.2. Application to neural networks. Consider the neural network (3.6) discussed in section 3.2. This is a system of coupled nonlinear equations, and we have seen that each subsystem is ISS. Note that so far we have not imposed any restrictions on the coefficients t_{ij}. Moreover, the assumptions imposed on a_i, b_i, and s_i are essentially milder than in [35]. However, to obtain the ISS property of the network, we need to require more. The small gain condition can be used for this purpose. It will impose restrictions on the coupling terms $t_{ij}s(x_j)$. From Corollary 5.6, Theorem 7.4 follows.

THEOREM 7.4. *Consider the Cohen–Grossberg neural network* (3.6)*. Let Γ_μ be given by γ_{ij} and μ_i, $i, j = 1, \ldots, n$, as calculated for the interconnection in section* 3.2*. Assume that Γ_μ satisfies the strong small gain condition $D \circ \Gamma_\mu \not\geq \mathrm{id}$ for $s \in \mathbb{R}^n_+ \setminus 0$. Then this network is ISS from $(J_1, \ldots, J_n)^T$ to x.*

Remark 7.5. In [35] the authors have proved that there exists a unique equilibrium point for the network and given constant external inputs. They have also proved the exponential stability of this equilibrium. We have considered arbitrary external inputs to the network and proved the ISS property for the interconnection.

8. Path construction. This section explains the relation between the small gain condition for Γ_μ and its mapping properties. Then we construct a strictly increasing Ω-path and prove Theorem 5.2 and some extensions. Let us first consider some simple particular cases to explain the main ideas, as depicted in Figure 8.1. In the following subsections we then proceed to the main path construction results.

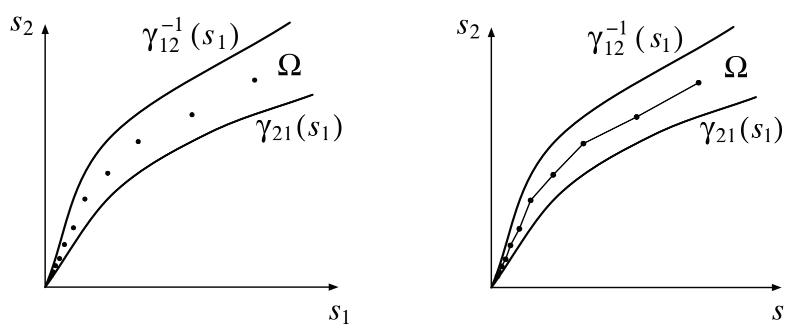

FIG. 8.1. *A sequence of points $\{\Gamma^k_\mu(s)\}_{k\geq 0}$ for some $s \in \Omega(\Gamma_\mu)$, where $\Gamma_\mu : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ is given by $\Gamma_\mu(s) = (\gamma_{12}(s_2), \gamma_{21}(s_1))^T$ and satisfies $\Gamma_\mu \not\geq \mathrm{id}$ or, equivalently, $\gamma_{21} \circ \gamma_{12} < \mathrm{id}$ and the corresponding linear interpolation; cf. Lemmas* 8.1, 8.2, *and* 8.3.

A map $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is *monotone* if $x \leq y$ implies $T(x) \leq T(y)$. Clearly, any matrix $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ together with an aggregation $\mu \in \mathrm{MAF}^n_n$ induces a monotone map Γ_μ.

LEMMA 8.1. *Let $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ and $\mu \in \mathrm{MAF}^n_n$ such that Γ_μ satisfies* (SGC)*. If $s \in \Omega(\Gamma_\mu)$, then $\lim_{k\to\infty} \Gamma^k_\mu(s) = 0$.*

Proof. If $s \in \Omega$, then $\Gamma_\mu(s) < s$, and by monotonicity $\Gamma^2_\mu(s) \leq \Gamma_\mu(s)$. By induction $\Gamma^k_\mu(s)$ is a monotonically decreasing sequence bounded from below by 0. Thus $\lim_{k\to\infty} \Gamma^k_\mu(s) =: s^*$ exists, and by continuity we have $\Gamma_\mu(s^*) = s^*$. By the small gain condition it follows that $s^* = 0$. ◻

LEMMA 8.2. *Assume that $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ has no zero rows, and let $\mu \in \mathrm{MAF}^n_n$. If $0 < s \in \Omega(\Gamma_\mu)$, then*

(i) *$0 < \Gamma_\mu(s) \in \Omega$;*

(ii) *for all $\lambda \in [0, 1]$ the convex combination $s_\lambda := \lambda s + (1 - \lambda)\Gamma_\mu(s) \in \Omega$.*

Proof. (i) By assumption $\Gamma_\mu(s) < s$, and so by the monotonicity assumption (M2), we have $\Gamma_\mu(\Gamma_\mu(s)) < \Gamma_\mu(s)$. Furthermore, as $s > 0$ and the matrix Γ has no zero rows, we have that $\Gamma_\mu(s) > 0$ by assumption (M1).

(ii) As $\Gamma_\mu(s) < s$ it follows for all $\lambda \in (0,1)$ that $\Gamma_\mu(s) < s_\lambda < s$. Hence by monotonicity and using (i),

$$0 < \Gamma_\mu(\Gamma_\mu(s)) < \Gamma_\mu(s_\lambda) < \Gamma_\mu(s) < s_\lambda\,.$$

This implies $s_\lambda \in \Omega$ as desired. ∎

LEMMA 8.3. *Assume that $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ has no zero rows, and let $\mu \in \mathrm{MAF}_n^n$ be such that Γ_μ satisfies* (SGC). *Let $s \in \Omega(\Gamma_\mu)$. Then there exists a path in $\Omega \cup \{0\}$ connecting the origin and s.*

Proof. By Lemma 8.2, the line segment $\{\lambda\Gamma_\mu(s) + (1-\lambda)s\} \subset \Omega$. By induction all the line segments $\{\lambda\Gamma_\mu^{k+1}(s) + (1-\lambda)\Gamma_\mu^k(s)\} \subset \Omega$ for $k \geq 1$. Using Lemma 8.1, we see that $\Gamma_\mu^k(s) \to 0$ as $k \to \infty$. This constructs an Ω-path with respect to Γ_μ from 0 to s. ∎

The following result applies to Γ whose entries are bounded, i.e., in $(\mathcal{K}\backslash\mathcal{K}_\infty)\cup\{0\}$.

PROPOSITION 8.4. *Assume that $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ has no zero rows, and let $\mu \in \mathrm{MAF}_n^n$ be such that Γ_μ satisfies* (SGC). *Assume furthermore that Γ_μ is bounded; then there exists an Ω-path with respect to Γ_μ.*

Proof. By assumption the set $\Gamma_\mu(\mathbb{R}_+^n)$ is bounded, so pick $s > \sup \Gamma_\mu(\mathbb{R}_+^n)$. Then clearly, $\Gamma_\mu(s) < s$ and so $s \in \Omega$. By the same argument $\eta s \in \Omega$ for all $\eta \in [1,\infty)$. Thus a path in Ω through the point s exists if we find a path from s to 0 contained in Ω. The remainder of the result is given by Lemma 8.3. ∎

The difficulty now arises if Γ_μ happens to be unbounded; i.e., Γ contains entries of class $\mathcal{K}_\infty$. In the unbounded case the simple construction above is not possible. In the following we will first consider the case that all nonzero entries of Γ are of class $\mathcal{K}_\infty$. Beforehand, we introduce a few technical lemmas.

8.1. Technical lemmas. Throughout this subsection $T : \mathbb{R}_+^n \to \mathbb{R}_+^n$ denotes a continuous, monotone map; i.e., T satisfies $T(v) \leq T(w)$ whenever $v \leq w$. We start with a few observations.

LEMMA 8.5. *Let $\rho \in \mathcal{K}_\infty$. Then there exists a $\tilde\rho \in \mathcal{K}_\infty$ such that $(\mathrm{id}+\rho)^{-1} = \mathrm{id}-\tilde\rho$.*

Proof. Just define $\tilde\rho = \rho\circ(\mathrm{id}+\rho)^{-1}$. Then $(\mathrm{id}-\tilde\rho)\circ(\mathrm{id}+\rho) = (\mathrm{id}+\rho)-\tilde\rho\circ(\mathrm{id}+\rho) = \mathrm{id} + \rho - \rho\circ(\mathrm{id}+\rho)^{-1}\circ(\mathrm{id}+\rho) = \mathrm{id} + \rho - \rho = \mathrm{id}$, which proves the lemma. ∎

LEMMA 8.6.

(i) *Let $D = \mathrm{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty$ such that $\rho > \mathrm{id}$. Then for any $k \geq 0$ there exist $\rho_1^{(k)}, \rho_2^{(k)} \in \mathcal{K}_\infty$, satisfying $\rho_i^{(k)} > \mathrm{id}$ such that for $D_i^{(k)} = \mathrm{diag}(\rho_i^{(k)})$, $i = 1,2$,*

$$D = D_1^{(k)} \circ D_2^{(k)}.$$

Moreover, $D_2^{(k)}$, $k \geq 0$, can be chosen such that for all $0 < s \in \mathbb{R}_+^n$ we have

$$D_2^{(k)}(s) < D_2^{(k+1)}(s).$$

(ii) *Let $D = \mathrm{diag}(\mathrm{id}+\alpha)$ for some $\alpha \in \mathcal{K}_\infty$. Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that for $D_i = \mathrm{diag}(\mathrm{id}+\alpha_i)$, $i = 1,2$,*

$$D = D_1 \circ D_2.$$

For maps $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ define the *decay set*

$$\Psi(T) := \{s \in \mathbb{R}^n_+ : T(s) \le s\}\,,$$

where we again omit the reference to T if this is clear from the context.

LEMMA 8.7. *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and $D = \operatorname{diag}(\rho)$ for some $\rho \in \mathcal{K}_\infty, \rho > \mathrm{id}$. Then*

(i) *$T^{k+1}(\Psi) \subset T^k(\Psi)$ for all $k \ge 0$;*

(ii) *$\Psi(D \circ T) \cap \{s \in \mathbb{R}^n_+ : s > 0\} \subset \Omega(T)$ if T satisfies $T(v) < T(w)$ whenever $v < w$; the same is true for $D \circ T$ replaced by $T \circ D$.*

The proofs of the lemmas are simple and thus omitted for reasons of space. Nevertheless, they can be found in [24, pp. 10, 29].

We will need the following connectedness property in the following.

PROPOSITION 8.8. *Let $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ and $\mu \in \mathrm{MAF}^n_n$ be such that Γ_μ satisfies* (SGC)*. Then Ψ is nonempty and pathwise connected. Moreover, if Γ_μ satisfies $\Gamma_\mu(v) < \Gamma_\mu(w)$ whenever $v < w$, then for any $s \in \Omega(\Gamma_\mu)$ there exists a strictly increasing Ω-path connecting 0 and s.*

Proof. Note that always $0 \in \Psi$; hence, Ψ cannot be empty. Along the lines of the proof of Lemma 8.3, it follows that each point in Ψ is pathwise connected to the origin. ∎

Another crucial step, which is of topological nature, regards preimages of points in the decay set Ψ. In general it is not guaranteed that for $s \in \mathbb{R}^n_+$ with $T(s) \in \Psi$, we also have $s \in \Psi$. The set of points in Ψ for which preimages of arbitrary order are also in Ψ is the set

$$\Psi_\infty(T) := \bigcap_{k=0}^{\infty} T^k(\Psi),$$

compare Figure 8.2. Of course, this set might be empty or bounded. We will use it to construct Ω-paths for operators Γ_μ satisfying the small gain condition.

PROPOSITION 8.9 (see [25, Proposition 5.4]). *Let $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be monotone and continuous and satisfy $T(s) \not\ge s$ for all $s \ne 0$. Assume that T satisfies the property*

$$\|s_k\| \to \infty \implies \|T(s_k)\| \longrightarrow \infty \tag{8.1}$$

as $k \to \infty$ for any sequence $\{s_k\}_{k\in\mathbb{N}} \subset \mathbb{R}^n_+$.

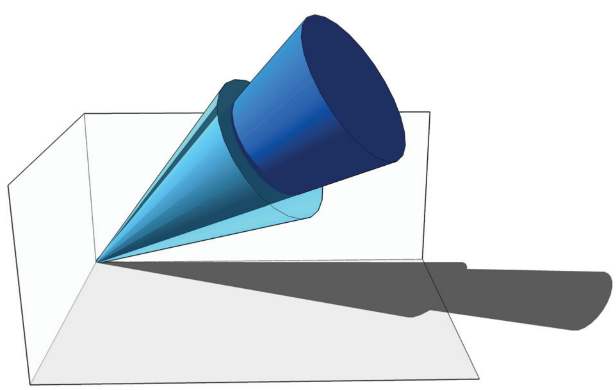

FIG. 8.2. *A sketch of the set $\Psi_\infty \subset \Psi \subset \mathbb{R}^n_+$ in Proposition* 8.9.

Then $\Psi_\infty(T) \subset \Psi(T)$, $\Psi_\infty(T) \cap S_r \neq \emptyset$ *for all* $r \geq 0$, *and* $\Psi_\infty(T)$ *is unbounded.*

A result based on the topological fixed point theorem from Knaster, Kuratowski, and Mazurkiewicz allows us to relate Ω and the small gain condition. It is essential for the proof of Proposition 8.9.

PROPOSITION 8.10. *Let* $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ *be monotone and continuous. If* $T(s) \not\geq s$ *for all* $s \in \mathbb{R}^n_+$, *then the set* $\Omega \cap S_r$ *is nonempty for all* $r > 0$.

In particular, $s \in \Omega \cap S_r$ for $r > 0$ implies $s > 0$. The proof for this result can be found in [24, Proposition 1.5.3, p. 26] or in a slightly different form in [6].

8.2. Paths for $\mathcal{K}_\infty \cup \{0\}$ gain matrices. In this subsection we consider matrices $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$; i.e., all nonzero entries of Γ are assumed to be unbounded functions.

In this setting we assume and utilize that the graph associated to Γ is strongly connected; i.e., Γ is irreducible. So if we consider powers $\Gamma^k_\mu(x)$ for each components i and j, there exists a $k = k(i,j)$ such that $t \mapsto \Gamma^k_\mu(t \cdot e_j)_i$ is an unbounded function.

THEOREM 8.11. *Let* $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ *be irreducible,* $\mu \in \mathrm{MAF}^n_n$, *and assume* $\Gamma_\mu \not\geq \mathrm{id}$. *Then there exists a strictly increasing path* $\sigma \in \mathcal{K}^n_\infty$ *satisfying*

$$\Gamma_\mu(\sigma(r)) < \sigma(r) \quad \textit{for all } r > 0.$$

The main technical difficulty in the proof is to construct the path in the unbounded direction; the other case has already been dealt with in Proposition 8.8.

The proof comprises the following steps: First, due to [25, Proposition 5.8], we may choose a $\mathcal{K}_\infty$-function $\varphi > \mathrm{id}$ so that for $D = \mathrm{diag}(\varphi)$ we have $\Gamma_\mu \circ D \not\geq \mathrm{id}$. Then we construct a monotone (but not necessarily strictly monotone) sequence $\{s^k\}_{k\geq 0}$ in $\Psi(\Gamma_\mu \circ D)$, satisfying $s^k = \Gamma_\mu(D(s^{k+1})) \lneqq s^{k+1}$, so that each component sequence is unbounded. At this point a linear interpolation of the sequence points may not yield a *strictly increasing* path. So finally we use the "extra space" provided by D in the set $\Omega(\Gamma_\mu) \supset \Omega(\Gamma_\mu \circ D)$ to obtain a strictly increasing sequence $\{\tilde{s}^k\}_{k\geq 0}$ in $\Omega(\Gamma_\mu)$ which we can linearly interpolate to obtain the desired Ω-path.

Proof. Since Γ is irreducible, it has no zero rows, and hence Γ_μ satisfies $\Gamma_\mu(v) < \Gamma_\mu(w)$ whenever $v < w$. By [25, Proposition 5.8] there exists a $\varphi > \mathrm{id}$ so that for $D = \mathrm{diag}(\varphi)$ we have $\Gamma_\mu \circ D \not\geq \mathrm{id}$. Now we construct a nondecreasing sequence $\{s^k\}$ in $\Psi(\Gamma_\mu \circ D)$.

Let $T := \Gamma_\mu \circ D$. Then T and by induction also all powers T^l, $l \geq 1$, satisfy (8.1).

By Proposition 8.9 the set $\Psi_\infty(T)$ is unbounded, so we may pick an $0 \neq s^0 \in \Psi_\infty(T)$. We can actually choose $s^0 > 0$, since the sequence $\{s^k\}$ we are going to construct will be unbounded in every component, at which point we may replace s^0 by some $s^k > 0$ for k large enough.

Due to irreducibility of Γ (and Remark 2.6), the following property holds: for any pair $1 \leq i,j \leq n$ there exists an $l \geq 1$ such that

$$r \mapsto (\Gamma^l_\mu(re_j))_i \tag{8.2}$$

is an unbounded and increasing function, where e_j is the jth unit vector. By monotonicity the same holds when T is considered instead of Γ_μ. Now define a sequence $\{s^k\}_{k\geq 0}$ by choosing

$$s^{k+1} \in T^{-1}(s^k) \cap \Psi_\infty(T)$$

for $k \geq 0$. This is possible, since by definition $\Psi_\infty(T)$ is backward invariant under T.

This sequence $\{s^k\}$ satisfies $s^k \lneqq s^{k+1}$ by definition. We claim that it is unbounded, and also unbounded in every component: to this end assume first that it is bounded. Then by monotonicity there exists a limit $s^* = \lim_{k\to\infty} s^k$. By continuity of T and since $s^k = T(s^{k+1})$, we have

$$s^* = \lim_{k\to\infty} s^k = \lim_{k\to\infty} T(s^{k+1}) = T\left(\lim_{k\to\infty} s^{k+1}\right) = T(s^*),$$

contradicting $T(s) \not\geq s$ for all $s \neq 0$. Hence the sequence $\{s^k\}$ must be unbounded.

Let j be an index such that $\{s_j^k\}_{k\in\mathbb{N}}$ is unbounded; let $i \in \{1,\dots,n\}$ be arbitrary, and choose l such that the function in (8.2) is unbounded for i, j, l. Choose real numbers $r_k \to \infty$ such that $r_k e_j \leq s^k$ for all $k \in \mathbb{N}$. Then we have

$$(T^l(r_k e_j))_i \leq (T^l(s^k))_i = s_i^{k-l}\,.$$

As the term on the left goes to ∞ for $k \to \infty$, so does s_i^k. Hence $\{s^k\}$ is unbounded in every component.

Now by Lemma 8.7(ii) the sequence $\{s^k\}$ is contained in $\Omega(\Gamma_\mu)$, but it may not be strictly increasing, as we only know $s^k \lneqq s^{k+1}$ for all $k \geq 0$. We define a strictly increasing sequence $\{\tilde{s}^k\}$ as follows: By Lemma 8.6 for any $k \geq 0$ we may factorize $D = D_1^{(k)} \circ D_2^{(k)}$ so that $D_1^{(k)}, D_2^{(k)} > \mathrm{id}$ and $D_2^{(k)}(s) < D_2^{(k+1)}(s)$ for all $k \geq 0$ and all $s > 0$. Using this factorization, we define

$$\tilde{s}^k := D_2^{(k)}(s^k)$$

for all $k \geq 0$. By the definition of $D_2^{(k)}$ this sequence is clearly strictly increasing and inherits from $\{s^k\}$ the unboundedness in all components.

We claim that $\{\tilde{s}^k\} \subset \Omega(\Gamma_\mu)$. This follows from

$$\tilde{s}^k > s^k \geq \Gamma_\mu \circ D(s^k) = \Gamma_\mu \circ D_1^{(k)} \circ D_2^{(k)}(s^k) = \Gamma_\mu \circ D_1^{(k)}(\tilde{s}^k) > \Gamma_\mu(\tilde{s}^k).$$

Now we prove that for $\lambda \in (0,1)$ we have $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$. Clearly,

$$\tilde{s}^k < (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} < \tilde{s}^{k+1},$$

and application of the strictly increasing operator Γ_μ yields

$$\begin{aligned}\Gamma_\mu((1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}) &< \Gamma_\mu(\tilde{s}^{k+1}) \\ &= \Gamma_\mu \circ D_2^{(k+1)}(s^{k+1}) < \Gamma_\mu \circ D_1^{(k+1)} \circ D_2^{(k+1)}(s^{k+1}) \\ &= s^k < \tilde{s}^k < (1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1}.\end{aligned}$$

Hence $(1-\lambda)\tilde{s}^k + \lambda\tilde{s}^{k+1} \in \Omega(\Gamma_\mu)$.

Now we may define σ as a parametrization of the linear interpolation of the points $\{\tilde{s}^k\}_{k\geq 0}$ in the unbounded direction and utilize the construction from Lemma 8.3 for the other direction. Clearly, this function σ is an Ω-path as it has component functions of class $\mathcal{K}_\infty$ and is piecewise linear on every compact interval contained in $(0,\infty)$. ∎

It is possible to consider the reducible case in a similar fashion. The argument is essentially an induction over the number of irreducible and zero blocks on the diagonal of the reducible operator. We cite the following result from [25, Theorem 5.10]. However, for the construction of an ISS Lyapunov function in the case of reducible Γ, we take a different route as described in section 6, thus avoiding the use of assumption (M4).

THEOREM 8.12. *Let* $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ *be reducible,* $\mu \in \mathrm{MAF}_n^n$ *satisfy* (M4), $D = \mathrm{diag}(\mathrm{id} + \alpha)$ *for some* $\rho \in \mathcal{K}_\infty$*, and assume* $\Gamma_\mu \circ D \not\geq \mathrm{id}$. *Then there exists a monotone and continuous operator* $\tilde{D} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ *and a strictly increasing path* $\sigma : \mathbb{R}_+ \to \mathbb{R}_+^n$ *whose component functions are all unbounded such that* $\Gamma_\mu \circ \tilde{D}(\sigma) < \sigma$.

8.3. General $\boldsymbol{\Gamma_\mu}$. In the preceding subsections we have seen that it is possible to construct Ω-paths for matrices Γ whose nonzero entries are either all bounded or all unbounded. It remains to consider the case that the nonzero entries of Γ are partly of class $\mathcal{K}_\infty$ and partly of class $\mathcal{K} \setminus \mathcal{K}_\infty$. We can state the following result.

PROPOSITION 8.13. *Let* $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$*, and let* $\mu \in \mathrm{MAF}_n^n$ *satisfy* (M4). *Assume* Γ_μ *satisfies* (sSGC). *Then there exists an* Ω*-path for* Γ_μ.

Proof. Write

$$\Gamma = \Gamma_U + \Gamma_B,$$

with $\Gamma_U \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ and $\Gamma_B \in (\mathcal{K} \setminus \mathcal{K}_\infty \cup \{0\})^{n\times n}$. Clearly, we have $(\Gamma_U)_\mu \leq \Gamma_\mu$ and $(\Gamma_B)_\mu \leq \Gamma_\mu$, and hence both maps satisfy

$$(\Gamma_\bullet)_\mu \not\geq \mathrm{id},$$

where $\bullet$ serves as a placeholder for the subscripts U and B.

The map $(\Gamma_B)_\mu$ is bounded. Hence $s^* := \sup(\Gamma_B)_\mu(\mathbb{R}_+^n)$ is a finite vector.

By Theorem 8.12 for $(\Gamma_U)_\mu$ there exists a $\mathcal{K}_\infty$-function $\tilde{\rho}$ and a $\mathcal{K}_\infty$-path σ_U so that for the diagonal operator $\tilde{D} = \mathrm{diag}(\mathrm{id} + \tilde{\rho})$ we have

$$((\Gamma_U)_\mu \circ \tilde{D})(\sigma_U(r)) < \sigma_U(r) \quad \text{for all } r > 0\,.$$

Similarly, by Proposition 8.4, there exists a $\mathcal{K}_\infty$-path σ_B such that $(\Gamma_B)_\mu(\sigma_B(r)) < \sigma_B(r)$ for all $r > 0$. In fact, and this is the key to this proof, it is possible to reparametrize σ_B in the region where $\sigma_B(r) > s^*$ as follows: for any $\alpha, \beta \in \mathcal{K}_\infty$ we can find a $\kappa \in \mathcal{K}_\infty$ such that

$$(\alpha \circ \kappa)(r) < \beta(r), \quad r > 0,$$

e.g., by choosing $\kappa \in \mathcal{K}_\infty$ satisfying $\kappa(r) < (\alpha^{-1} \circ \beta)(r)$. This is always possible. Denote $\bar{D} = \mathrm{diag}(\tilde{\rho})$ (so that $\tilde{D} = \mathrm{id} + \bar{D}$), and choose r^* such that $\bar{D}(\sigma_U(r^*)) > s^*$. Then after reparametrization we may assume that

$$\sigma_B(r) < \bar{D}(\sigma_U(r)) \quad \text{and} \quad \sigma_B(r) > s^*$$

for all $r \geq r^*$. Using Lemma 8.3, we let $\sigma_L : [0, r^*] \to \mathbb{R}_+^n$ be a finite-length path satisfying

$$\begin{gathered} \Gamma_\mu(\sigma_L(r)) < \sigma_L(r) \quad \text{for all } r \in (0, r^*], \\ \sigma_L \text{ is strictly increasing,} \\ \sigma_L(0) = 0, \text{ and } \sigma_L(r^*) = \sigma_B(r^*) + \sigma_U(r^*). \end{gathered}$$

Now define σ by

$$\sigma(r) = \begin{cases} \sigma_B(r) + \sigma_U(r) & \text{if } r > r^*, \\ \sigma_L(r) & \text{if } r < r^*. \end{cases}$$

It remains to check that σ satisfies $\Gamma_\mu(\sigma(r)) < \sigma(r)$ for $r \geq r^*$. Indeed, for $r \geq r^*$ we have

$$\begin{aligned}\sigma(r) &= \sigma_U(r) + \sigma_B(r) > ((\Gamma_U)_\mu \circ \tilde{D})(\sigma_U(r)) + s^* \\ &> (\Gamma_U)_\mu(\sigma_U(r) + \sigma_B(r)) + (\Gamma_B)_\mu(\sigma_U(r) + \sigma_B(r)) \\ &\geq \Gamma_\mu(\sigma_U(r) + \sigma_B(r)),\end{aligned}$$

where the last inequality is due to (M4). This completes the proof. ◻

8.4. Special case: Maximization. The case when the aggregation is the maximum (i.e., $\mu = \max$) is indeed a special case, since not only the small gain condition can be formulated in simpler manner but also the path construction can be achieved without the need of the diagonal operator D as before.

A *cycle* in a matrix Γ is a finite sequence of nonzero entries of Γ of the form

$$(\gamma_{i_1,i_2}, \gamma_{i_2,i_3}, \dots, \gamma_{i_K,i_1}).$$

A cycle is called *subordinated* if $i_1 > \max\{i_2, \dots, i_K\}$, and it is called a *contraction* if

$$\gamma_{i_1,i_2} \circ \gamma_{i_2,i_3} \circ \dots \circ \gamma_{i_K,i_1} < \mathrm{id}.$$

It is an easy exercise to show that when all subordinated cycles are contractions, then already all cycles are contractions.

THEOREM 8.14. *Let $\mu = \max$ and $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$. If all subordinated cycles of Γ are contractions, then there exists an Ω-path with respect to Γ_μ.*

The proof is composed of the following steps. The first step is to show that the cycle condition (all cycles being contractions) is equivalent to $\Gamma_\mu \not\geq \mathrm{id}$. Note that $\mu = \max$ automatically satisfies (M4), but (M4) is actually not needed for the proof. Then the path construction can essentially be done as before, replacing sums by maximization, and one can even avoid the use of $D = \mathrm{diag}(\mathrm{id} + \rho)$. Compare also [25].

8.5. Proof of Theorem 5.2. We now come to the easiest part of this section, which is to combine all the preceding results to one general theorem for matrices with entries of class $\mathcal{K}$, namely, Theorem 5.2.

Proof of Theorem 5.2.

(i) In the linear case we can identify Γ_μ with a real matrix with nonnegative entries. Then there exists a positive vector $v > 0$ so that $\Gamma_\mu v < v$ if the spectral radius $\rho(\Gamma_\mu) < 1$; cf. [2] or [24, Lemma 2.0.1, p. 33]. For $r > 0$ this gives $\Gamma_\mu rv < rv$; i.e., a $\mathcal{K}_\infty$-path is given by $\sigma(r) = rv$.

(ii) This is Theorem 8.11.

(iii) This is Theorem 8.14.

(iv) This is Proposition 8.4. ◻

9. Remarks for the case of three subsystems. Recall that a construction of an Ω-path σ for the case of two subsystems was given in [17]. We have seen that in a general case of $n \in \mathbb{N}$ subsystems the construction involves more theory and topological properties of Γ_μ that follow from the small gain condition. However, in the case of three subsystems σ can be found by rather simple considerations. Here we provide this illustrative construction. Let us consider the special case $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{3\times 3}$, $\mu_i(s) = s_1 + s_2 + s_3$, $i = 1, 2, 3$, and for simplicity assume that $\gamma_{ij} \in \mathcal{K}_\infty$ for all $i \neq j$ so that

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{bmatrix}, \quad \Gamma_\mu(s) = \begin{pmatrix} \gamma_{12}(s_2) + \gamma_{13}(s_3) \\ \gamma_{21}(s_1) + \gamma_{23}(s_3) \\ \gamma_{31}(s_1) + \gamma_{32}(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix}. \tag{9.1}$$

Fix $s_1 \geq 0$; then it follows that there is exactly one s_2 satisfying

$$\gamma_{13}^{-1}(s_1 - \gamma_{12}(s_2)) = \gamma_{23}^{-1}(s_2 - \gamma_{21}(s_1))\,, \tag{9.2}$$

since for a fixed s_1 the left side of (9.2) is a strictly decreasing function of s_2 while the right side of (9.2) is a strictly increasing one. The small gain condition (9.1), in particular, ensures that $\gamma_{12}^{-1}(\gamma_{21}^{-1}(r)) > r$ for any $r > 0$. Let s_2^* be the solution of $s_1 - \gamma_{12}(s_2) = 0$ and s_2^{**} be the solution of $s_2 - \gamma_{21}(s_1) = 0$; then

$$s_2^* = \gamma_{12}^{-1}(s_1) = \gamma_{12}^{-1}(\gamma_{21}^{-1}(s_2^{**})) > s_2^{**}.$$

Hence the root of the left side of (9.2) is greater than the root of the right side of (9.2). This proves that for any s_1 there is always exactly one s_2 satisfying (9.2); see Figure 9.1.

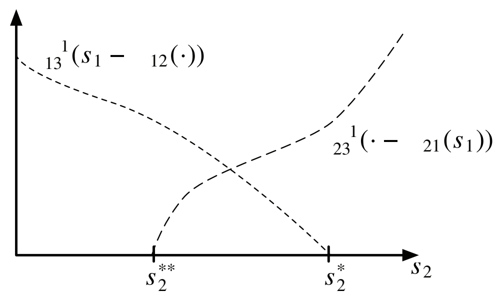

FIG. 9.1. *Visualization of* (9.2).

By the continuity and monotonicity of $\gamma_{12}, \gamma_{21}, \gamma_{13}$, and γ_{23}, it follows that s_2 depends continuously on s_1 and is strictly increasing with s_1. We can define $\sigma_1(r) = r$ for $r \geq 0$ and $\sigma_2(r)$ to be the unique s_2 solving (9.2) for $s_1 = r$.

Denote $h(r) = \gamma_{31}(\sigma_1(r)) + \gamma_{32}(\sigma_2(r))$ and $g(r) = \gamma_{13}^{-1}(\sigma_1(r) - \gamma_{12}(\sigma_2(r))) = \gamma_{23}^{-1}(\sigma_2(r) - \gamma_{21}(\sigma_1(r)))$, and define $M(r) := \{s_3 : h(r) < s_3 < g(r)\}$. Let us show that $M(r) \neq \emptyset$ for all $r > 0$. If this is not true, then there exists $r^* > 0$ such that $s_3^* := h(r^*) \geq g(r^*)$ holds. Consider the point $s^* := (s_1^*, s_2^*, s_3^*) := (r^*, \sigma_2(r^*), s_3^*)$. Then $s_3^* \geq g(r^*) = \gamma_{13}^{-1}(s_1^* - \gamma_{12}(s_2^*))$, $s_3^* \geq g(r^*) = \gamma_{23}^{-1}(s_2^* - \gamma_{21}(s_1^*))$, and $s_3^* = h(r^*) = \gamma_{31}(s_1^*) + \gamma_{32}(s_2^*)$. In other words,

$$\Gamma(s^*) = \begin{pmatrix} \gamma_{12}(s_2^*) + \gamma_{13}(s_3^*) \\ \gamma_{21}(s_1^*) + \gamma_{23}(s_3^*) \\ \gamma_{31}(s_1^*) + \gamma_{32}(s_2^*) \end{pmatrix} \geq \begin{pmatrix} s_1^* \\ s_2^* \\ s_3^* \end{pmatrix}\,,$$

contradicting (2.1). Hence $M(r)$ is not empty for all $r > 0$.

Consider the functions $h(r)$ and $g(r)$. The question is how to choose $\sigma_3(r) \in M(r)$ such that $\sigma_3 \in \mathcal{K}_\infty$. Note that $h(r) \in \mathcal{K}_\infty$. Let $g^*(r) := \min_{u \geq r} g(u)$ so that $g^*(r) \leq g(r)$ for all $r \geq 0$. Since $h(r)$ is unbounded, for all $r > 0$ the set $C(r) := \arg\min_{u \geq r} g(u)$ is compact, and for all points $p \in C(r)$ the relation $g^*(r) \geq g(p) > h(p) \geq h(r)$ holds. We have $h(r) < g^*(r) \leq g(r)$ for all $r > 0$, where g^* is a (not necessarily strictly) increasing function. Now take $\sigma_3(r) := \frac{1}{2}(g^*(r) + h(r))$, and observe that $\sigma_3 \in \mathcal{K}_\infty$ and $h(r) < \sigma_3(r) < g^*(r)$ for all $r > 0$. Hence $\sigma := (\sigma_1, \sigma_2, \sigma_3)^T$ satisfies $\Gamma_\mu(\sigma(r)) < \sigma(r)$ for all $r > 0$.

The case where one of γ_{ij}'s is not a $\mathcal{K}_\infty$-function but zero can be treated similarly.

10. Conclusions. In this paper we have provided a method for the construction of ISS Lyapunov functions for interconnections of nonlinear ISS systems. The method applies for an interconnection of an arbitrary finite number of subsystems interconnected in an arbitrary way and satisfying a small gain condition. The small gain condition is imposed on the nonlinear gain operator Γ_μ that we have introduced here. This operator contains the information of the topological structure of the network and the interactions between its subsystems. An ISS Lyapunov function for such a network is given in terms of ISS Lyapunov functions of subsystems and some auxiliary functions. We have shown how this construction is related to the small gain condition and mapping properties of the gain operator Γ_μ and its invariant sets. Namely, the small gain condition guarantees the existence of an unbounded vector function with its path in an invariant set Ω of the operator Γ_μ. This auxiliary function can be used to rescale the ISS Lyapunov functions of the individual subsystems and aggregate them into an ISS Lyapunov function for the entire network. The construction technique for this vector function has been detailed as well as the construction of the composite Lyapunov function. The constructed Lyapunov function is only locally Lipschitz continuous so that methods from nonsmooth analysis had to be used. The proposed method has been exemplified for linear systems and neural networks.

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