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Local ISS of large-scale interconnections and estimates for stability regions

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ABSTRACT

We consider interconnections of locally input-to-state stable (LISS) systems. The class of LISS systems is quite large, in particular it contains input-to-state stable (ISS) and integral input-to-state stable (iISS) systems.

Local small-gain conditions both for LISS trajectory and Lyapunov formulations guaranteeing LISS of the composite system are provided in this paper. Notably, estimates for the resulting stability region of the composite system are also given. This in particular provides an advantage over the linearization approach, as will be discussed.

1. Introduction

In this paper we study local stability properties of interconnected nonlinear systems. One of the most popular frameworks for such interconnections is input-to-state stability (ISS) introduced in [1]. This notion has been used successfully for the investigation of continuous and discrete time systems, systems with time delays, and hybrid systems. In particular the first small-gain stability condition for a feedback interconnection of two ISS systems which were given in terms of ordinary differential equations was derived in [2]. A corresponding construction of an ISS Lyapunov function for feedback interconnections has been given in [3]. These results were extended for the case of an interconnection of $n \geq 2$ systems in [4–6], respectively. Small-gain theorems for hybrid systems can be found in [7,8]. Interconnected systems with time delays have been studied in the ISS framework in [9]. A small-gain theorem for interconnections of a more general type of systems that do not satisfy the classical semigroup property has been developed in [10].

In some applications the ISS property can be rather restrictive. A less restrictive property is for example the integral input-to-state stability (iISS) property [11]. The set of iISS systems contains ISS systems as a proper subset. Small-gain theorems for interconnections of iISS systems can be found in [12,13]. Another way to weaken the ISS property is to consider its local version, local input-to-state stability (LISS), but see also [14–16] for different local stability properties. It turns out that LISS constitutes an even bigger class of nonlinear systems than iISS systems (cf. [17, Theorem 1]: iISS implies 0-GAS and [16, Lemma I.1]: 0-GAS implies LISS). In broad terms, a system is LISS if the ISS property holds locally with respect to inputs and initial states. Systems with such restrictions and a corresponding small-gain condition for feedback interconnections of two systems have been discussed in [2]. Large-scale interconnections of such systems have been considered in [18] for the first time.

Provided that the stability regions of allowable inputs and initial conditions are quantified and suitably large, LISS is a rather interesting property from an application perspective, as it allows to estimate transient and asymptotic behavior of solutions of nonlinear systems in a well-understood framework.

This paper is devoted to stability investigations of large-scale interconnected nonlinear systems. To this extent, we consider $n \geq 2$ subsystems given by

$$\dot{x}_i = f_i(x_1, \dots, x_n, u_i), \quad i = 1, \dots, n, \tag{1}$$

where $x_i \in \mathbb{R}^{N_i}$, $u_i \in \mathbb{R}^{M_i}$, and $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \dots, n$, are assumed to be continuous and locally Lipschitz in x uniformly for u_i in compact sets, which guarantees existence (at least on small time intervals) and uniqueness of solution for each of the systems. Let x^T denote the transposition of a vector. Introducing $x^T = (x_1^T, \dots, x_n^T) \in \mathbb{R}^N$, $N = \sum_{i=1}^n N_i$, $M = \sum_{i=1}^n M_i$, $u^T = (u_1^T, \dots, u_n^T)$, $f(x,u)^T = (f_1(x,u_1)^T, \dots, f_n(x,u_n)^T)$ we consider

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this interconnection as one composite system of a larger dimension N,

$$\dot{x} = f(x, u). \quad (2)$$

Our main results are small-gain theorems that provide sufficient conditions for the stability of such interconnections: Under the assumption that each system (1) is LISS (see below) and a small-gain condition, we show that the composite system (2) is also LISS.

In particular, we provide a local small-gain condition, which turns out to be similar but weaker than its global counterpart in [4–6]. We also show how the Lyapunov functions of the subsystems can be aggregated to a composite Lyapunov function. The approach is similar and heavily inspired by its global counterpart; however, a number of technical modifications are in order, which will be provided. Most notably and in contrast to previous works and existing literature based on linearization, our results provide estimates on the regions where the stability results hold. In addition, by utilizing the concept proposed in [16], our results also apply to stability with respect to sets, rather than just equilibrium points.

The paper is organized as follows. The next section introduces the necessary notions and formally states the problem. In Section 3 we recall corresponding global results for the stronger ISS property. Our local small-gain condition is introduced in Section 4 where we also prove some auxiliary results related to this condition. Section 5 contains the main results of the paper. In Section 5.3 we briefly highlight the advantages of LISS compared to linearization approaches. An illustrative example is considered in Section 6. Section 7 concludes the paper.

2. Notation and definitions, problem formulation

2.1. Notation

Let $\mathbb{R}^n_+ := \{x \in \mathbb{R}^n : x_i \geq 0,\ i = 1, \dots, n\}$ denote the positive orthant in $\mathbb{R}^n$. For $a, b \in \mathbb{R}^n_+$ let $a \ll b$ denote that $a_i < b_i$ for all $i = 1, \dots, n$ and $a \leq b$ denote $a_i \leq b_i$ for all $i = 1, \dots, n$. We write $a < b$ iff $a \leq b$ and $a \neq b$. With respect to this partial order, the minimum and maximum of two or more vectors is taken component-wise. For a vector $a \in \mathbb{R}^n$ by $|a|$ we denote the vector $(|a_1|, \dots, |a_n|)^T \in \mathbb{R}^n_+$. Observe that $|a| = \max\{a, -a\}$. The logical negation of the relation $\geq$ is denoted by $a \not\geq b$ and it means that there is at least one $i \in \{1, \dots, n\}$ such that $a_i < b_i$. It is *not* the same as the relation $<$. For $a, b \in \mathbb{R}^n_+$ we write $[a, b] := \{s \in \mathbb{R}^n_+ : a \leq s \leq b\}$, $(a, b) := \{s \in \mathbb{R}^n_+ : a < s < b\}$, and similarly $[a, b)$, $(a, b]$ to denote order intervals in $\mathbb{R}^n_+$. By $\|x\|$ we denote the Euclidean norm of $x \in \mathbb{R}^n$ and by $\|u\|_{L_\infty(T)} = \text{ess.}\sup_{t\in T} \|u(t)\|$ we denote the essential supremum norm of a measurable function u. Reference to the time interval T is usually omitted in the case $T = \mathbb{R}_+$. The set of all measurable and essentially bounded functions is denoted by L_∞. By $B(x, r)$ we denote the open ball with respect to the Euclidean norm around x of radius r. Let $\mathcal{A}$ be a nonempty set in $\mathbb{R}^n$. Then by $\|x\|_{\mathcal{A}} = d(x, \mathcal{A}) = \inf_{y\in\mathcal{A}} \|x - y\|$ we denote the distance between x and $\mathcal{A}$, cf. [16]. The induced L_∞-distance is denoted by $\|x\|_{L^{\mathcal{A}}_\infty(T)} := \text{ess.}\sup_{t\in T} \|x(t)\|_{\mathcal{A}}$.

A continuous operator $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is called monotone, if $r \leq s$ implies $A(r) \leq A(s)$. For a vector $x \in \mathbb{R}^n_+$ we denote by $x|_I$ the vector in $\mathbb{R}^n_+$ with elements

$$(x|_I)_i = \begin{cases} x_i & \text{if } i \in I \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded. We will frequently use the class $\mathcal{K}_\infty$ notation for functions that are defined only on bounded intervals $[0, r]$. In this case the function will obviously be bounded; however, it can always be extended to a $\mathcal{K}_\infty$ function on $[0, \infty)$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero at infinity.

2.2. Local input-to-state stability (LISS)

The concept of input-to-state stability (ISS) has been first introduced in [1]. Its local version, also with respect to a nonempty, compact set $\mathcal{A}$, has first appeared in [16].

Throughout let $\mathcal{A} \subset \mathbb{R}^N$ be nonempty, compact, and *zero-invariant* with respect to (2), i.e., $x(t, \xi, \underline{0}) \in \mathcal{A}$ for all $t \geq 0, \xi \in \mathcal{A}$, where $\underline{0}$ denotes the input which is identically zero and $x()$ denotes the unique solution to (2).

Definition 2.1. System (2) is *locally input-to-state stable* (LISS) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, and $\beta \in \mathcal{KL}$, such that for all $\|\xi\|_{\mathcal{A}} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\|x(t, \xi, u)\|_{\mathcal{A}} \leq \beta(\|\xi\|_{\mathcal{A}}, t) + \gamma(\|u\|_{L_\infty}), \quad \forall\, t \geq 0. \quad (3)$$

Here γ is called *LISS gain*.

If $\rho^0 = \rho^u = \infty$, then system (2) is called *input-to-state stable* (ISS) with respect to $\mathcal{A}$. It is known that ISS defined this way is equivalent to the existence of an ISS Lyapunov function. Here we give the definition of a LISS Lyapunov function:

Definition 2.2. A smooth function $V : \mathcal{D} \to \mathbb{R}_+$, with $\mathcal{D} \subset \mathbb{R}^N$ open, is a *LISS Lyapunov function* of (2) if there exist $\rho^0 > 0$, $\rho^u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}_\infty$, and a positive definite function α such that $B(0, \rho^0) \subset \mathcal{D}$ and

$$\psi_1(\|x\|_{\mathcal{A}}) \leq V(x) \leq \psi_2(\|x\|_{\mathcal{A}}), \quad \forall\, x \in \mathbb{R}^N, \quad (4)$$

$$V(x) \geq \gamma(\|u\|) \Longrightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)), \quad (5)$$

for all $\|x\|_{\mathcal{A}} \leq \rho^0$, $\|u\| \leq \rho^u$. The function γ is called *LISS Lyapunov gain*. If $\rho^0 = \rho^u = \infty$ then V is called an *ISS Lyapunov function*.

A related and strictly weaker stability concept (just think of the scalar system $\dot{x} = 0$) is that of local stability:

Definition 2.3. System (2) is *locally stable* (LS) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\sigma, \gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\|_{\mathcal{A}} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\operatorname*{ess.\,sup}_{t\geq 0} \|x(t, \xi, u)\|_{\mathcal{A}} \leq \sigma(\|\xi\|_{\mathcal{A}}) + \gamma(\|u\|_{L_\infty}). \quad (6)$$

Also related is the concept of asymptotic gains.

Definition 2.4. System (2) has the *local asymptotic gain property* (LAG) with respect to $\mathcal{A}$, if there exist $\rho^0 > 0$, $\rho^u > 0$, $\gamma \in \mathcal{K}_\infty$, such that for all $\|\xi\|_{\mathcal{A}} \leq \rho^0$, $\|u\|_{L_\infty} \leq \rho^u$

$$\limsup_{t\to\infty} \|x(t, \xi, u)\|_{\mathcal{A}} \leq \gamma(\|u\|_{L_\infty}). \quad (7)$$

Note that inequality (7) is equivalent to

$$\limsup_{t\to\infty} \|x(t, \xi, u)\|_{\mathcal{A}} \leq \gamma(\text{ess.}\limsup_{t\to\infty} \|u\|). \quad (8)$$

In all of the above stability definitions, the reference to $\mathcal{A}$ is usually omitted when $\mathcal{A} = \{0\}$.

2.3. Monotone aggregation functions (MAFs)

The concept of monotone aggregation functions has been introduced in [19] and has subsequently been used in [20,6,21]. It is useful to cover different formulations for the aggregation of multiple inputs in a unified way. Examples of MAFs include all monotone norms on $\mathbb{R}^n_+$ (which includes all p-norms).

Definition 2.5 (*Monotone Aggregation Functions*). A function $\mu : \mathbb{R}^n_+ \to \mathbb{R}_+$ is called a *monotone aggregation function* (MAF_n) if μ is continuous and

(M1) nonnegative: $\mu(s) \geq 0$ for all $s \in \mathbb{R}^n_+$;
(M2) strictly increasing: if $x \ll y$ then $\mu(x) < \mu(y)$.

By $\mu \in MAF^m_n$ we denote vector monotone aggregation functions, i.e., $\mu_i \in MAF_n$ for $i = 1, \dots, m$, and if $m = n$ we simply write MAF^n instead of MAF^n_n.

A direct consequence of (M2) and continuity is that also the weaker monotonicity property $x \leq y \implies \mu(x) \leq \mu(y)$ holds for MAFs.

Further assumptions made for global results in [6,21] include the properties

(M3) unboundedness: if $\|x\| \to \infty$ then $\mu(x) \to \infty$;
(M4) sub-additivity: $\mu(x + y) \leq \mu(x) + \mu(y)$.

Note that while we will require (M4) for a strict subset of our main results, none of the results derived in this paper assumes (M3). Standard examples satisfying (M1)–(M4) are summation and maximization, which we write as

$$\Sigma : (x_1, \dots, x_n)^T \mapsto \sum_{i=1}^{n} x_i, \quad \text{and}$$

$$\oplus : (x_1, \dots, x_n)^T \mapsto \max_{i=1,\dots,n} x_i.$$

The induced vector MAFs will be denoted by the same symbols.

2.4. LISS for multiple inputs and gain matrices

The stability notions in Section 2.2 can be extended to the case of several inputs to one subsystem as in (1), where x_j, for $j \neq i$, is regarded as an independent input to the ith subsystem. This is possible for both, the trajectory formulation as well as the Lyapunov formulation. Here we settle some unifying notation.

We assume that there exists a nonempty, compact set $\mathcal{A}_i \subset \mathbb{R}^{N_i}$, zero invariant with respect to the ith subsystem.

We call the ith subsystem LISS, provided there exist $\rho^0_i > 0$, $\rho^j_i > 0$, $\rho^u_i > 0$ and functions $\gamma_{ij}, \gamma_{iu} \in (\mathcal{K}_\infty \cup \{0\})$, $\beta_i \in \mathcal{KL}$, and a monotone aggregation function $\mu_i \in MAF_{n+1}$, such that for all $\xi_i \in \mathbb{R}^{N_i}$ such that $\|\xi_i\|_{\mathcal{A}_i} \leq \rho^0_i$, for all $x_j \in L_\infty(\mathbb{R}_+; \mathbb{R}^{N_j})$ such that $\|x_j\|_{L^{\mathcal{A}_j}_\infty} \leq \rho^j_i$ (where $j \neq i$), and for all $u_i \in L_\infty(\mathbb{R}_+; \mathbb{R}^{M_i})$ such that $\|u_i\|_{L_\infty} \leq \rho^u_i$, the following estimate holds for all $t \geq 0$:

$$\begin{aligned}&\|x_i(t; \xi_i, x_j : j \neq i, u_i)\|_{\mathcal{A}_i}\\ &\leq \beta_i(\|\xi_i\|_{\mathcal{A}_i}, t) + \mu_i\Big(\gamma_{i1}(\|x_1\|_{L^{\mathcal{A}_1}_\infty([0,t])}), \dots,\\ &\gamma_{in}(\|x_n\|_{L^{\mathcal{A}_n}_\infty([0,t])}), \gamma_{iu}(\|u_i\|_{L_\infty([0,t])})\Big).\end{aligned} \tag{9}$$

Similarly, for the Lyapunov formulation of LISS we have in the case of several inputs the following extension of Definition 2.2: A smooth function $V_i : \mathcal{D}_i \to \mathbb{R}_+$, $\mathcal{D}_i \subset \mathbb{R}^{N_i}$ open, such that for some $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$,

$$\psi_{i1}(\|x_i\|_{\mathcal{A}_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{\mathcal{A}_i}), \quad \forall\, x_i \in \mathcal{D}_i, \tag{10}$$

is a LISS Lyapunov function for subsystem (1) if there exist $\rho^0_i > 0$, $\rho^j_i > 0$, $\rho^u_i > 0$, functions $\gamma_{ij}, \gamma_{iu} \in \mathcal{K}_\infty \cup \{0\}$, a positive definite function α_i, and a monotone aggregation function $\mu_i \in MAF_{n+1}$, such that $B(0, \rho^0_i) \subset \mathcal{D}_i$ and for all $x_i \in \mathbb{R}^{N_i}$ with $\|x_i\|_{\mathcal{A}_i} < \rho^0_i$ and inputs satisfying $\|x_j\|_{\mathcal{A}_j} < \rho^j_i$ for $j \neq i$ and $u_i \in \mathbb{R}^{M_i}$, $\|u_i\| < \rho^u_i$, the following implication holds:

$$\begin{aligned}&V_i(x) \geq \mu_i\left(\gamma_{i1}(V_1(x_1)), \dots, \gamma_{in}(V_n(x_n)), \gamma_{iu}(\|u_i\|)\right)\\ &\implies \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -\alpha_i(V_i(x_i)).\end{aligned} \tag{11}$$

If all n subsystems in (1) are LISS, we can collect the gains in a matrix

$$\Gamma = \left(\gamma_{ij}\right)^n_{i,j=1}, \quad \text{with } \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\} \tag{12}$$

where we use the convention that $\gamma_{ii} \equiv 0$ for $i = 1, \dots, n$. Similarly, we collect the external gains γ_{iu} in a column vector $\Gamma^e(s) = (\gamma_{1u}(s_1), \dots, \gamma_{nu}(s_n))^T$.

The matrix Γ is called *gain matrix* of the interconnection (1). Note that $\gamma_{ij} \equiv 0$ means that there is no input from system j to system i, i.e., f_i does not depend on x_j.

The gain matrix Γ together with $\mu = (\mu_1, \dots, \mu_n)^T$ defines monotone operators, denoted by the symbols $\Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ and $\overline{\Gamma}_\mu : \mathbb{R}^{n+1}_+ \to \mathbb{R}^n_+$, given by

$$\Gamma_\mu(s)_i := \mu_i\left(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), 0\right) \tag{13}$$

for $s \in \mathbb{R}^n_+$ and $\overline{\Gamma}_\mu(s)_i := \mu_i\left(\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), \gamma_{iu}(s_{n+1})\right)$ for $s \in \mathbb{R}^{n+1}_+$.

Throughout the paper we make the following assumption to rule out pathological cases that might otherwise occur when we use this notation:

Assumption 2.6 (*Compatibility Assumption*). Given $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ and $\mu \in MAF^n$, we will from now on assume that Γ and μ are *compatible* in the following sense: For each $i = 1, \dots, n$, let I_i denote the set of indices corresponding to the nonzero entries in the ith row of Γ. Then it is understood that also the restriction of μ_i to the indices I_i satisfies (M2), i.e., if $x|_{I_i} \ll y|_{I_i}$ then $\mu_i(x|_{I_i}) < \mu_i(y|_{I_i})$.

All standard examples of MAFs, i.e., monotone norms including Σ and $\oplus$ satisfy this assumption.

2.5. A vector formulation for trajectory LISS estimates

For the trajectory formulation (9), we introduce a shorthand vector notation building upon gain operators:

We abbreviate $\|x_i(t)\|_{\mathcal{A}_i}$ by $s_i(t)$, $\|x_i\|_{L^{\mathcal{A}_i}_\infty([0,t])}$ by $s_{i,[0,t]}$ and form corresponding vectors $s(t) = (s_1(t), \dots, s_n(t))^T$ and $s_{[0,t]} = (s_{1,[0,t]}, \dots, s_{n,[0,t]})^T$. Analogously, by $e_{i,[0,t]}$ we refer to $\|u_i\|_{L_\infty([0,t])}$ and by $e_{[0,t]}$ to the corresponding vector.

For $v \in \mathbb{R}^n_+$ and $t \in \mathbb{R}_+$ let us write

$$B(v, t) = (\beta_1(v_1, t), \dots, \beta_n(v_n, t))^T.$$

To be able to give estimates for the stability regions, we define the vector notation $\boldsymbol{\rho}^i = (\rho^i_1, \dots, \rho^i_n)^T$, for $i = 0, \dots, n$, and $\boldsymbol{\rho}^e = (\rho^u_1, \dots, \rho^u_n)^T$. We also define $\boldsymbol{\rho}^x := \min_{i=1,\dots,n} \boldsymbol{\rho}^i$. Using this newly defined notation, the LISS estimates (9) for $i = 1, \dots, n$ can be written in vectorized form as follows:

Subsystems (1) are LISS for $i = 1, \dots, n$ if there exist vectors $\boldsymbol{\rho}^0, \boldsymbol{\rho}^x, \boldsymbol{\rho}^e \in \mathbb{R}^n_+$, $\boldsymbol{\rho}^0, \boldsymbol{\rho}^x, \boldsymbol{\rho}^e \gg 0$, such that for all $s(0) \ll \boldsymbol{\rho}^0$, $t \geq 0$, and the corresponding solutions and inputs to (1) satisfying $s_{[0,t]} \ll \boldsymbol{\rho}^x$ and $e_{[0,t]} \ll \boldsymbol{\rho}^e$, the following estimate holds:

$$s(t) \leq B(s(0), t) + \mu\left([\Gamma(s_{[0,t]}), \Gamma^e(e_{[0,t]})]\right). \tag{14}$$

If μ satisfies M4, then estimate (14) implies

$$s(t) \leq B(s(0), t) + \Gamma_\mu(s_{[0,t]}) + \Gamma^e(e_{[0,t]}). \tag{15}$$

We observe that we have $s(0) \leq s_{[0,t]}$ for all $t \geq 0$ and hence without loss of generality we may assume that $\rho^0 \leq \rho^x$.

Also note that in general $\|x\|_{L^{\mathcal{A}}_\infty([0,T])} \neq \|s_{[0,T]}\|$, e.g., for the case $\mathcal{A} = \{0\}$, $x(t) = (\cos t, \sin t)^T$ with $s_1(t) = |\cos t|$ and $s_2(t) = |\sin t|$. Here we find $\|x\|_{L^{\mathcal{A}}_\infty([0,2\pi])} = 1$, while $\|s_{[0,2\pi]}\| = \sqrt{2}$. For the Euclidean norm we have the following estimate:

Lemma 2.7. *For the above defined notation in general it holds that*

$$\|x\|_{L_\infty([0,T])} \leq \|s_{[0,T]}\| \leq \sqrt{n}\|x\|_{L_\infty([0,T])}.$$

3. Known global results

The global small-gain condition assuring the ISS property for an interconnection of $n \geq 2$ ISS systems has first been derived in [4]. An alternative proof has been given in [5]. We quote the following result from these papers for comparison.

Theorem 3.1. *Consider system (2) and suppose that each subsystem (1) is ISS, i.e., condition (9) holds for all $\xi_i \in \mathbb{R}^n_+$, $u_i \in L_\infty$, and $i = 1, \ldots, n$. Let Γ be given by (12) and let the monotone aggregation be Σ. If there exists an $\alpha \in \mathcal{K}_\infty$, such that*

$$(\Gamma_\Sigma \circ D)(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\}, \tag{16}$$

with $D = \operatorname{diag}_n(\mathrm{id} + \alpha)$ then the system (2) is ISS from u to x.

A version of this result for general μ satisfying M4 follows along the lines of the same proofs using the result [21, Theorem 6.1].

Furthermore it is known that under the same small-gain condition an ISS Lyapunov function for (2) can be explicitly constructed as a combination of the ISS Lyapunov functions of subsystems, see, e.g., [6, Corollary 5.5]:

Theorem 3.2. *Consider the interconnected systems (1), where each of the subsystems Σ_i is assumed to have an ISS Lyapunov function V_i and the corresponding gain matrix is given by (12). Assume that each $\mu_i \in MAF_{n+1}$ satisfies* (M3) *and is additive in the last argument, i.e.,*

$$\mu_i(s, r) = \mu_i(s, 0) + r, \quad \text{for all } s \in \mathbb{R}^n_+,\ r \in \mathbb{R}_+. \tag{17}$$

If Γ_μ is irreducible and if there exists an $\alpha \in \mathcal{K}_\infty$ such that for $D = \operatorname{diag}(\mathrm{id} + \alpha)$ the gain operator Γ_μ satisfies the condition

$$D \circ \Gamma_\mu(s) \not\geq s \tag{18}$$

then the interconnected system is ISS and there exists a vector valued function $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$ satisfying $(\Gamma \circ D)(\sigma(r)) \ll \sigma(r), \forall r > 0$, such that each component function σ_i is of class $\mathcal{K}_\infty$ and piecewise linear on $(0, \infty)$. Moreover,

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$$

is a nonsmooth ISS Lyapunov function for the system (2).

Note that $V(x)$ in this case is not smooth but only Lipschitz continuous. In [6] it has been pointed out that a Lipschitz continuous ISS Lyapunov function is sufficient to deduce input-to-state stability. The argument is the same for LISS Lyapunov functions.

A local version of the function σ can be constructed explicitly as we will show below.

4. Local small-gain condition

Motivated by the global small-gain conditions we introduce its local counterpart as follows. We say that Γ_μ satisfies the *local small-gain condition* on $[0, w^*]$, provided that

$$\Gamma_\mu(w^*) \ll w^* \quad \text{and} \quad \Gamma_\mu(s) \not\geq s, \quad \forall s \in [0, w^*],\ s \neq 0. \tag{19}$$

In this paper we give local results similar to Theorems 3.1 and 3.2. The following lemmas will be used to obtain the main result. We start with a simple criterion that guarantees (19).

Lemma 4.1. *Let Γ be a gain matrix as in (12), $\mu \in MAF^n$, and let $w^* \in \mathbb{R}^n_+$ satisfy $\Gamma_\mu(w^*) \ll w^*$. Consider the trajectory $\{w(k)\}$ of the discrete monotone system $w(k+1) = \Gamma_\mu(w(k))$, $k = 0, 1, 2 \ldots$ with $w(0) = w^*$. Then $w(k) \to 0$ for $k \to \infty$ if and only if Γ_μ satisfies the small-gain condition (19) on $[0, w^*]$.*

Remark 4.2. The previous lemma gives an idea how the small-gain condition (19) can be verified. First one looks for $w^* \in \mathbb{R}^n$ with $\Gamma_\mu(w^*) \ll w^*$. Then instead of checking $\Gamma_\mu(s) \not\geq s$ for all $s \in [0, w^*] \setminus \{0\}$ one needs to check whether the sequence $w(k)$ converges to the origin — and the latter is quite an easy task. For the first task, i.e., finding a suitable w^*, there exist numerical algorithms which can be efficiently implemented. See, e.g., [19, Chapter 4].

Proof. To prove sufficiency let $w(k) \to 0$ for $k \to \infty$ and suppose there exists a point $v \in [0, w^*]$ with

$$\Gamma_\mu(v) \geq v \tag{20}$$

and $v \neq 0$. Since Γ_μ is monotone, so is Γ^k_μ, i.e., the k-times application of Γ_μ. Hence (20) implies $\Gamma^k_\mu(v) \geq v \geq 0$, so $\Gamma^k_\mu(v)$ does not tend to zero as k approaches infinity. But $v \leq w^*$ implies $\Gamma^k_\mu(v) \leq \Gamma^k_\mu(w^*) = w(k)$, which is assumed to tend to zero. This contradiction implies that $\Gamma_\mu(v) \not\geq v$ for all $v \in [0, w^*]$ and sufficiency is proved.

Now assume that Γ_μ satisfies (19) on $[0, w^*]$ and consider the sequence $\{w(k)\}$, $k = 0, 1, \ldots$ defined by $w(k+1) = \Gamma_\mu(w(k))$ and $w(0) = w^*$. By the monotonicity of Γ_μ and $\Gamma_\mu(w^*) \ll w^*$ this sequence is bounded in $\mathbb{R}^n$ and hence it contains a convergent subsequence that converges to some $v \in \mathbb{R}^n$. Then by the continuity of Γ_μ we have $\Gamma_\mu(v) = v$ contradicting (19) unless $v = 0$. This proves the necessity. □

For the case that Γ has no zero rows and Γ_μ satisfies (19), the result [21, Proposition 5.2] shows that a linear interpolation of the sequence $\{\Gamma^k_\mu(w^*)\}_{k\geq 0}$ gives a path, called an *Ω-path*, satisfying

$$\sigma : [0, 1] \to [0, w^*], \qquad \Gamma_\mu(\sigma(r)) \ll \sigma(r), \quad \text{for } r \in (0, 1], \tag{21}$$

and that each component function σ_i is strictly increasing. Furthermore, σ is piecewise linear and satisfies $\sigma(0) = 0$, $\sigma(1) = w^*$. An extension is the following result.

Proposition 4.3. *Let $\Gamma_\mu : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be given by (13). Assume that Γ has no zero row and satisfies (19). Then there exists a path $\sigma : [0, 1] \to [0, w^*]$ and a function $\varphi \in \mathcal{K}_\infty$ such that*

$$\overline{\Gamma}_\mu(\sigma(r), \varphi(r)) \ll \sigma(r), \quad \text{for all } r \in (0, 1]. \tag{22}$$

Proof. Let σ be given by [21, Proposition 5.2], i.e., as a linear interpolation of the sequence points

$$\{\Gamma^k_\mu(w^*)\}_{k\geq 0}, \tag{23}$$

so that σ satisfies (21). For each $\rho \in (0, 1]$ let $\phi_r = \sup\{s > 0 : \overline{\Gamma}_\mu(\sigma(r), s) \ll \sigma(r)\} \in [0, \infty]$. By monotonicity and continuity of $\overline{\Gamma}_\mu$ we have $\phi_r > 0$, $\forall r \in (0, 1]$, so we may take any $\varphi \in \mathcal{K}_\infty$ satisfying $\varphi(r) < \phi_r$, $\forall r \in (0, 1]$. □

The following result is a local version of the main ingredient which has been used in [5] to prove the global ISS small-gain theorem.

Proposition 4.4. *Let $w^* \in \mathbb{R}^n_+$, $w^* \gg 0$. Let Γ_μ given by (13) satisfy (19) on $[0, w^*]$. Assume Γ has no zero row. Assume μ satisfies M4. Then for each $w' \in (\Gamma_\mu(w^*), w^*)$ there exists a function $\varphi \in \mathcal{K}_\infty$, such that for all $w \in \mathbb{R}^n_+$, $0 \le w \le w'$, and all $v \in \mathbb{R}^n_+$ we have*

$$(\mathrm{id} - \Gamma_\mu)(w) \le v \Longrightarrow w \le \mathrm{diag}(\varphi)(v). \tag{24}$$

Proof. This proof essentially goes along the lines of the proof of [5, Lemma 13] (which requires use of (M4)), once it is established that there exists an operator $D = \mathrm{diag}(\mathrm{id} + \kappa)$, $\kappa \in \mathcal{K}_\infty$, such that $D \circ \Gamma_\mu(s) \not\ge s$ for all $s \ne 0$, $s \in [0, w']$. The existence of this operator D can be guaranteed using essentially the same technical construction as in [21, Proposition 5.8]. Combining all these ingredients, one obtains the estimate $w \le \min\{w', \mathrm{diag}(\mathrm{id} + \kappa^{-1})^n(v)\}$. From here we can conclude $w_i < (\mathrm{id} + \kappa^{-1})^n(v_i)$ if v_i is small. So we may take $\varphi(r) = (\mathrm{id} + \kappa^{-1})^n(r)$. □

Remark 4.5. Note that choice of a "large" φ in Proposition 4.4 leads to strong restrictions on initial conditions and inputs that one gets for the whole interconnection to be stable, see Remark 5.4. However in some cases this is inevitable (for example, if Γ_μ is close to the identity by its mapping properties).

5. Main results

In the first two of the following subsections we assume that each subsystem (1) is LISS with respect to $\mathcal{A}_i$ and show that the local small-gain condition (19) is sufficient to imply LISS of the composite system (2) with respect to $\mathcal{A} = \mathcal{A}_1 \times \cdots \times \mathcal{A}_n$. Furthermore, we show how a LISS Lyapunov function can be constructed. Notably, in both cases estimates for the stability region are provided.

By linearizing the gain operator, the sufficient conditions for interconnection stability can be simplified at the expense of the estimates on the stability regions that one would otherwise obtain, see Section 5.4.

5.1. A local small-gain theorem

In this subsection we state a small-gain theorem for large-scale interconnected systems comprised of LISS subsystems in the spirit of [4,5].

Theorem 5.1. *Let all subsystems (1), $i = 1, \ldots, n$, satisfy (9). Suppose Γ_μ satisfies (19) and Γ has no zero rows. Then there exist $\rho^0 > 0$, $\rho^u > 0$, $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}_\infty$, such that system (2) satisfies (3) with respect to the set $\mathcal{A}$.*

Remark 5.2. Observe that the inequality $\Gamma_\mu \not\ge \mathrm{id}$ in the small-gain condition (19) is weaker than the one in (16), since the latter additionally contains the operator D. Recall from [5, Example 18] that D is in fact essential to assure the global ISS property. However, in case of local ISS this D can be omitted.

The proof consists of two steps: First we establish that system (2) is LS and LAG with respect to $\mathcal{A}$, then by a result due to Sontag and Wang [16] LISS with respect to $\mathcal{A}$ follows.

Proof of Theorem 5.1. For brevity we use the notation of Section 2.5, i.e., we assume that (14), (15) hold. Throughout the proof let $w' \in (\Gamma_\mu(w^*), w^*)$ be fixed and φ be given by Proposition 4.4. Denote $F := \mathrm{diag}(\varphi)$. Let $\boldsymbol{\varepsilon} := \min\{\boldsymbol{\rho}^x, \boldsymbol{\rho}^0, w'\} \in \mathbb{R}^n_+$ and $\boldsymbol{\delta} := F^{-1}(\frac{1}{2}\boldsymbol{\varepsilon}) \in \mathbb{R}^n_+$.

Step 1 — Existence of solutions and local stability. Let $u \in L_\infty(\mathbb{R}_+; \mathbb{R}^M)$ such that $e_{[0,\infty]} = e_{[0,\infty]}(u)$ satisfies $e_{[0,\infty]} \ll \boldsymbol{\rho}^e$ and $\Gamma^e(e_{[0,\infty]}) \ll \frac{1}{2}\boldsymbol{\delta}$. Consider initial states $x(0) = \xi \in \mathbb{R}^N$ such that $s(0) = s(0; \xi, u) \in \mathbb{R}^n$ satisfies $B(s(0), 0) \ll \frac{1}{2}\boldsymbol{\delta}$ and $s(0) \ll \boldsymbol{\varepsilon}$.

Define $T^* := \inf\{t \ge 0 : s(t) = s(t; \xi, u) \not\le \boldsymbol{\varepsilon}\}$. Clearly $s_{[0,T^*]} \le \boldsymbol{\varepsilon}$ and hence $s_{[0,T^*]} \le w'$. So we may apply Proposition 4.4 to the following inequality:

$$(\mathrm{id} - \Gamma_\mu)(s_{[0,T^*]}) \le B(s(0), 0) + \Gamma^e(e_{[0,\infty]}).$$

Hence $s_{[0,T^*]} \le F\left(B(s(0), 0) + \Gamma^e(e_{[0,\infty]})\right) \ll F(\boldsymbol{\delta}/2 + \boldsymbol{\delta}/2) = \frac{1}{2}\boldsymbol{\varepsilon}$. This implies that there is no finite minimal time T^*, such that the component-wise distance from the trajectory $x(\cdot, \xi, u)$ to the set $\mathcal{A}$ leaves an $\boldsymbol{\varepsilon}$-neighborhood. Hence this trajectory stays in that open set for all times.

Now let $\rho^0 := \sup\{\|s\| : s \in \mathbb{R}^n_+, s \le \boldsymbol{\varepsilon}, B(s, 0) \ll \frac{1}{2}\boldsymbol{\delta}\}$ and choose $0 < \rho^u < \sup\{\|e\| : e \in \mathbb{R}^n_+, e \le \boldsymbol{\rho}^e, \Gamma^e(e) \le \frac{1}{2}\boldsymbol{\delta}\}$. Then it follows that for $\|\xi\|_{\mathcal{A}} < \rho^0$ and $\|u\|_{L_\infty} \le \rho^u$ the solution $x(\cdot; \xi, u)$ exists for all times and is bounded in $\|\cdot\|_{\mathcal{A}}$-distance by $\frac{1}{2}\|\boldsymbol{\varepsilon}\|$. In fact, we have (using Lemma 2.7), the weak triangle inequality [2] and the norm triangle inequality,

$$\begin{aligned}\operatorname*{ess\,sup}_{t \ge 0} \|x(t; \xi, u)\|_{\mathcal{A}} &\le \|F(B(s(0), 0) + \Gamma^e(e_{[0,\infty]}))\| \\ &\le \|F(2B(s(0), 0))\| + \|F(2\Gamma^e(e_{[0,\infty]}))\| \\ &\le \sigma(\|\xi\|_{\mathcal{A}}) + \gamma(\|u\|_{L_\infty})\end{aligned}$$

for some $\sigma, \gamma \in \mathcal{K}_\infty$. This establishes LS with respect to $\mathcal{A}$.

Step 2 — An estimate of the form (8) can be established using essentially the same steps as in the proof of [5, Theorem 9].

Now by [16, Theorem 1:(f), (g)] it follows that (2) is LISS with respect to $\mathcal{A}$. □

Remark 5.3 (*An Alternative to the $\mathcal{KL}$-Estimate*). Instead of constructing the $\mathcal{KL}$-estimate in the last step of the preceding proof we could have argued that the set $\mathcal{A}$ is locally asymptotically stable with respect to the composite, externally unforced system $\dot{x} = f(x, 0)$. Following the lines of [16, Lemma I.2] (a result showing that 0-GAS implies LISS) and thereby using [22, Theorem 14*] for a suitable converse Lyapunov result, local input-to-state stability could also be shown this way. See also [23, Theorem 1] for a similar result for hybrid systems.

Remark 5.4 (*Stability Regions*). In the proof we have obtained

$$\rho^0 := \sup\left\{\|s\| : s \in \mathbb{R}^n_+, s \le \boldsymbol{\varepsilon}, B(s, 0) \ll \frac{1}{2}\boldsymbol{\delta}\right\}$$

and, essentially,

$$\rho^u = \sup\left\{\|e\| : e \in \mathbb{R}^n_+, e \le \boldsymbol{\rho}^e, \Gamma^e(e) \le \frac{1}{2}\boldsymbol{\delta}\right\},$$

where $\boldsymbol{\varepsilon} = \min\{\boldsymbol{\rho}^x, \boldsymbol{\rho}^0, w'\} \in \mathbb{R}^n_+$ and $\boldsymbol{\delta} = F^{-1}(\frac{1}{2}\boldsymbol{\varepsilon}) \in \mathbb{R}^n_+$. Here w' had to be chosen in the open order interval $(\Gamma_\mu(w^*), w^*)$ and $F = \mathrm{diag}(\varphi)$, with φ given by Proposition 4.4.

5.2. A small-gain theorem using LISS Lyapunov functions

In this section we assume that each subsystem i of (1) is LISS and admits a LISS Lyapunov function V_i with the corresponding gains γ_{ij}, γ_{iu}, MAFs μ_i and corresponding Γ_μ so that the implication (11) holds for all $\|x_i\|_{\mathcal{A}_i} < \rho_i^0$ and $x \in \mathbb{R}^N$ with $\|x_j\|_{\mathcal{A}_j} < \rho_i^j$ for $j \neq i$ and $u_i \in \mathbb{R}^{M_i}$ with $\|u_i\| < \rho_i^u$. We are looking for explicit expressions for the restrictions on the states x and inputs u such that the overall system is LISS.

First of all let us say that vectors $x = (x_1^T, \dots, x_n^T)^T \in \mathbb{R}^N$ with $\|x_i\|_{\mathcal{A}_i} \geq \rho_i^0$ or $\|x_i\|_{\mathcal{A}_j} \geq \rho_i^j$ for at least one i are not of interest, because the implication (11) is not available in this case. Similarly, any x with $\|x_i\|_{\mathcal{A}_i} > w_i^*$ is out of interest since then the small-gain condition does not apply. Hence a necessary restriction on states is already given in terms of ρ_i^0, w^* and $\boldsymbol{\rho}^x := (\min_j \rho_1^j, \dots, \min_j \rho_n^j)^T$. In some cases these restrictions are already enough, see Corollary 5.6, for the LISS property of the interconnection provided the local small-gain condition is satisfied. In general we have the following result.

Theorem 5.5. *Assume that each system i of the interconnection (1) is LISS and that it admits a LISS Lyapunov function V_i satisfying (11) for all $\|x_i\|_{\mathcal{A}_i} < \rho_i^0$, $\|x_j\|_{\mathcal{A}_j} < \rho_i^j$, and $\|u_i\| < \rho_i^u$. Let Γ and μ be given by (11) and assume Γ has no zero rows. Assume Γ_μ satisfies (19). Then the composite system (2) is LISS with respect to $\mathcal{A} = \prod \mathcal{A}_i$. Moreover, a nonsmooth LISS Lyapunov function with respect to the set $\mathcal{A}$ is given by*

$$V(x) = \max_i \sigma_i^{-1}(V_i(x_i)) \tag{25}$$

where σ is given by Proposition 4.3. Moreover, with this Lyapunov function V implication (5) holds for all $x \in \mathbb{R}^N$ such that $\|x_i\|_{\mathcal{A}_i} < \widetilde{\boldsymbol{\rho}}_i$ for all i, with $\widetilde{\boldsymbol{\rho}}_i := \min\{\boldsymbol{\rho}_i^x, \rho_i^0, w_i^*, \psi_{i2}^{-1}(w_i^*)\}$ and $u \in B_{\mathbb{R}^{M_1}}(0, \rho_1^u) \times \dots \times B_{\mathbb{R}^{M_n}}(0, \rho_n^u)$.

Proof. By Proposition 4.3 the local small-gain condition implies the existence of strictly increasing functions $\sigma_i : [0, 1] \to [0, w_i^*]$, such that $\sigma = (\sigma_1, \dots, \sigma_n)^T$ satisfies (22). Note that $\sigma_i^{-1} : [0, w_i^*] \to [0, 1]$ is well defined and such that for any compact set $K \subset (0, \infty)$ there exist $c, C > 0$ such that $c < (\sigma_i^{-1})' < C$. We define $V(x)$ by (25). To assure that $\sigma_i^{-1}(V_i(x_i))$ is well defined we have required $\|x_i\|_{\mathcal{A}_i} < \psi_{i2}^{-1}(w_i^*)$ which implies $V_i(x_i) < w_i^*$ by (10).

The proof that $V(x)$ is a LISS Lyapunov function for the interconnection follows along the lines of the proof of Theorem 5.3 in [6]. □

Corollary 5.6. *Consider interconnection (1) such that each system i has the same properties as in Theorem 5.5. If $V_i(x_i) \leq \|x_i\|_{\mathcal{A}_i}$ holds for all i then the assertion of Theorem 5.5 holds for all $x \in \mathbb{R}^N$ such that $\|x_i\|_{\mathcal{A}_i} < \min\{\boldsymbol{\rho}_i^x, \rho_i^0, w_i^*\}$ for all i and $u \in B_{\mathbb{R}^{M_1}}(0, \rho_1^u) \times \dots \times B_{\mathbb{R}^{M_n}}(0, \rho_n^u)$.*

This follows from the fact that ψ_{i2} can be taken to be the identity in Theorem 5.5. The corollary shows that in this case the restriction on initial values and external inputs depends essentially only on w^* and certainly on the restrictions initially given for the subsystems.

5.3. Advantages of LISS vs. linearization of systems

Since LISS is a local stability concept we would like to briefly discuss its advantages compared with local assertions obtained by linearization. To study stability of nonlinear systems as well as their interconnections one can resort to linearization — and stability results for interconnections of linear systems are well known. However, all information on stability obtained this way holds locally, where locally means that there exists some domain around the point of linearization where a certain stability property holds. Usually, there is no information about the size of this stability domain. In addition, linearization is only applicable to consider stability of equilibrium points, not sets.

The advantage in using the LISS property in considering interconnection stability is the possibility to obtain or at least to estimate the regions of stability. The present results yield information on how large initial conditions or external inputs may be taken to produce stable behavior. Furthermore, such estimates also work in the case when one considers invariant sets rather than equilibrium points. To provide explicit expressions for the stability regions of the interconnection in such cases is the main motivation behind this paper.

5.4. LISS small-gain theorems and linearization of gains

An important connection of the LISS approach to linear stability theory arises of course, when the subject of the stability condition happens to be linear, or, for that matter, can be linearized.

Assume given a gain operator $\Gamma_\mu : \mathbb{R}_+^n \to \mathbb{R}_+^n$ we have its Jacobian matrix at zero, denoted by $J\Gamma_\mu(0) \in \mathbb{R}^{n\times n}$, at our disposal. Clearly the elements of $J\Gamma_\mu(0)$ have to be nonnegative, and the diagonal will have only zero entries. For any nonnegative matrix $G \in \mathbb{R}_+^{n\times n}$ let $\rho(G) = \max\{|\lambda|; \lambda$ is an eigenvalue of $G\}$ denote the spectral radius of G. Then the following statements are equivalent:

1. G satisfies $\rho(G) < 1$;
2. $G^k \to 0$ as $k \to \infty$;
3. the matrix $(I - G)^{-1}$ exists and is nonnegative;
4. there exists a positive vector $s \in \mathbb{R}_+^n$, $s \gg 0$, such that $Gs \ll s$.

For a proof see, e.g., [21, Lemma 1.1]. Notable is also the fact that

$$\rho(G) < 1 \iff \exists D = \operatorname{diag}(1 + \boldsymbol{\varepsilon}), \quad \boldsymbol{\varepsilon} > 0 : \rho(DG) < 1,$$

where D is a linear version of the robustness operator which appeared in (16) and (18).

In this case $Gs \ll s$ implies that the Ω-path can be taken linear, by defining $\sigma(r) = sr \in \mathbb{R}_+^n$. From here the function φ can be constructed as before in Proposition 4.3. The composite Lyapunov function in Theorem 5.5 then takes the simple form

$$V(x) = \max_i V_i(x_i)/s_i. \tag{26}$$

Similarly, for the trajectory result Theorem 5.1, the inverse $(I - J\Gamma_\mu(0))^{-1}$ can be used directly instead of estimate (24). To summarize, we obtain the following corollaries to Theorems 5.1 and 5.5, respectively, both for the case that $\mathcal{A}_i = \{0\}$ for all i.

Corollary 5.7. *Let all subsystems (1), $i = 1, \dots, n$, satisfy (9). Suppose Γ_μ is differentiable at $s = 0$ and assume Γ has no zero rows. If $\rho(J\Gamma_\mu(0)) < 1$ then system (2) is LISS.*

Corollary 5.8. *Assume that each system i of the interconnection (1) is LISS and that it admits a LISS Lyapunov function V_i satisfying (11) for all $\|x_i\| < \rho_i^0$, $\|x_j\| < \rho_i^j$, and $\|u_i\| < \rho_i^u$. Let Γ and μ be given by (11). Assume the Γ_μ is differentiable at zero and the Jacobian matrix $J\Gamma_\mu(0)$ satisfies $\rho(J\Gamma_\mu(0)) < 1$. Then the composite system (2) is LISS and a nonsmooth LISS Lyapunov function is given by (26).*

6. Example

The following example illustrates the use of MAFs and the application of the main result Theorem 5.5. Consider the following system of n coupled equations with subsystems given by

$$\dot{x}_i(t) = -(n+1)x_i + x_i^2 + \sum_{\substack{j=1\\ j\neq i}}^{n} x_j + u_i,$$
$$x_i, u_i \in \mathbb{R},\ i = 1, \dots, n. \tag{27}$$

Consider the ith subsystem. Obviously, because of the presence of the quadratic term, this system is not 0-GAS, since for zero inputs and a large initial states $x_i(0)$ the trajectories are unbounded. In particular, none of the subsystem is ISS. However, it can be shown that each subsystem is LISS with $\rho_i^0 = \frac{n}{2}$, $\rho_i^j = 1-\varepsilon$ and $\rho_i^u = 1-\varepsilon$ for arbitrary small $\varepsilon > 0$ as follows. To this end consider $V_i(x_i) := |x_i|$ as a LISS Lyapunov function candidate. We define $\gamma_{ij} := \mathrm{id}$ for $i \neq j$, $\gamma_{iu} = \mathrm{id}$ and $\mu_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ by $\mu_i(s_1, \dots, s_{n+1}) := \frac{1}{2}\left(n+1+\varepsilon - \sqrt{(n+1+\varepsilon)^2 - 4\sum_{\substack{j=1\\ j\neq i}}^{n+1} s_j}\right)$. Note that μ_i satisfies M1 and M2 as well as the compatibility assumption for $s_j \leq 1-\varepsilon, j = 1, \dots, n+1$.

Now if $V_i(x_i) = |x_i| > \mu_i(\gamma_{i1}(|x_1|), \dots, \gamma_{in}(|x_n|), \gamma_{iu}(|u_i|))$ it follows that $\dot{V}_i(x_i) < -(n+1)|x_i| + |x_i|^2 + \sum_{j=1}^{n} |x_j| + |u_i| < -\varepsilon(|x_i|)$. This shows that each subsystem is LISS.

Since the subsystems are only LISS, but not ISS, we cannot apply the global result, Theorem 3.1, or its Lyapunov counterpart Theorem 3.2, but we have to resort to the local results developed in this paper.

Let $x := (x_1, \dots, x_n)^T$ and $u := (u_1, \dots, u_n)^T$. Let $w^* := (1-\varepsilon, \dots, 1-\varepsilon)^T \in \mathbb{R}^n$. It is easy to check that $\Gamma_\mu(w^*) \ll w^*$ and that the local small-gain condition is satisfied for this w^*. Let $\sigma_i = (1-\varepsilon)\,\mathrm{id}$ for $i = 1, \dots, n$ and $\sigma = (\sigma_1, \dots, \sigma_n)^T$ and $\varphi = (1-\varepsilon)\,\mathrm{id}$. Choosing ε small enough it can be checked that for any $\tau \in (0, 1]$ it holds $\overline{\Gamma}_\mu(\sigma(\tau), \varphi(\tau)) \ll (\sigma(\tau))$. Hence by Corollary 5.6 we conclude that the interconnection is LISS with LISS Lyapunov function $V(x) := \max_i |x_i|$ and subject to the following restrictions on $x, u \in \mathbb{R}^n$: $|x_i| < 1-\varepsilon$ for $i = 1, \dots, n$ and $|u_i| < 1-\varepsilon$, i.e., $\rho^0 = \rho^u = 1-\varepsilon$ for some small positive number ε.

Note that an alternative approach would be to consider instead an interconnection of the linearized versions of subsystems (27). A special case of the global results in Section 3 for linear systems (cf. [5, Corollary 19]) or the more general and precise results in [24] can then be used to assure that the composite *linear system* is globally asymptotically stable, which in turn suggests that the composite nonlinear system is at least locally asymptotically stable. However, this approach does not provide any information about the size of the local stability region for the nonlinear composite system or any information about robustness of this stability with respect to external disturbances. In contrast, Corollary 5.6 does provide this type of information.

7. Conclusions

We have presented a new nonlinear tool for the stability assessment of nonlinear interconnected systems, which complements the existing linearization theory. Our approach allows to consider general interconnections of locally input-to-state stable systems, a class much larger than ISS systems which also includes integral input-to-state stable systems. The results presented cover trajectory estimates as well as a Lyapunov version. Most notably, we also provide a constructive approach to aggregate Lyapunov functions of subsystems into a composite Lyapunov function. A nontrivial example illustrates how this method works. In contrast to the linear theory, the LISS approach readily provides estimates for the stability regions.

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