Local input to state stability of production networks

Sergey Dashkovskiy, Michael Görges, Lars Naujok

Abstract In this paper we analyze a given production network in view of stability, which means boundedness of the state of the network over time. By a mathematical point of view we will model the system by differential equations. This results in the derivation of conditions for which the production network is stable.

1 Introduction

Production and supply networks or other modern logistic structures are typical examples of complex systems with a nonlinear and sometimes chaotic behavior. Their dynamics subject to many different perturbations due to changes on market, changes in customer behavior, information and transport congestions, unreliable elements of the network etc.

One approach to handle such complex systems is to shift from centralized to decentralized or autonomous control, i.e. to allow the entities of a network to make their own decisions based on some given rules and the available local information. However a system emerging in this way may be not effective in performance or become unstable. Hence it is worth to investigate its behavior in advance.

Mathematical methods can help to handle these complex systems. In particular mathematical modelling and analysis provide helpful tools for investigation of such objects and can be used for design, optimization and control of such networks and for deeper understanding of their dynamical properties.

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One of the main properties that a logistic systems should satisfy is stability. In particular cases this property means that the number of the unsatisfied orders or/and amount of workload to be processed by a machine remain bounded over time in spite of disturbances. Obviously this property is decisive for the performance and vitality of a network.

In this paper we propose a model for a production logistic scenario comprising several autonomous production plants connected through transport routs. This network is modelled by ordinary autonomous differential equations. We show how its stability can by analyzed with help of small gain theorems recently developed for general type of dynamic networks. Explicit conditions of the production rates will be derived by application of mathematical systems theory of interconnected systems.

In Section 2 we describe the given production network with its parameters and model it mathematically by differential equations. A mathematical background is given in Section 3, which is used in Section 4 to derive stability conditions of the production network. In Section 5 some simulation results and their interpretations are given. Conclusions and outlines can be found in Section 6.

2 Model description

In this section we describe the given production network, which we are going to analyze in view of stability. We model the system with help of differential equations.

The production network in Figure 1 consists of six different production locations. The *state* of each production location is denoted by $x_i(t) \in \mathbb{R}_+$ for $i = 1, \ldots, 6$, where $t \in \mathbb{R}_+$ can be interpreted as time and $\mathbb{R}_+$ denotes all positive real values. In the rest of this paper we write *subsystem i* for the i-th production location and interpret the state of the i-th subsystem as the number of unsatisfied orders within the i-th production location. All six subsystems form the production network, which we name simply *(whole) system*. The arrows in Figure 1 describe the material flow. We will describe the production network by the information flow. Subsystem six gets some orders of its product from the customers, denoting by $d(t) \in \mathbb{R}_+$. While processing the orders, subsystem six sends orders for components, which they need for production to subsystems four and five. These two subsystems send orders for components, which they need for production to subsystems two and three. Their

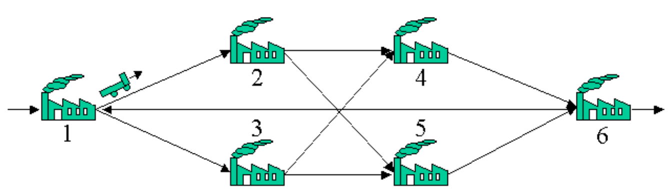

Fig. 1 The production network

orders will be sent to subsystem one, which gets instantly its raw material from an external source.

The orders from subsystem one to subsystem six can be interpreted as a payment or the demand for its production of subsystem one of the final product of the given production network from subsystem six.

We suppose all subsystems are autonomously controlled, it means that the change of the actual production rate at time t can be interpreted as the ability to vary the production rate of the production location. For example this could be varying work times of the workers, transportation times of the products or the number of used machines for production. The actual production rate of each subsystem at time t is given by

$$\tilde{f}_i(x_i(t)) := \alpha_i (1 - \exp(-x_i(t))), \; i = 1, \ldots, 6,$$

where $\alpha_i \in \mathbb{R}_+$ is the (constant) maximum production rate of subsystem i. $\tilde{f}_i$ converges to α_i, if the state of subsystem $x_i(t)$ is large and $\tilde{f}_i$ tends to zero, if the state of subsystem $x_i(t)$ tends to zero. This means, if there are many orders, the actual production rate is near to the maximum production rate and if there are no orders nothing will be produced.

With this considerations we can model the system presented in Figure 1 by differential equations for each subsystem, which are nothing but a description of changes of the state $x_i(t)$ of subsystem i along time $t \in \mathbb{R}_+$:

$$\begin{aligned}
\dot{x}_1(t) &= c_{21}\tilde{f}_2(x_2(t)) + c_{31}\tilde{f}_3(x_3(t)) - \tilde{f}_1(x_1(t)),\\
\dot{x}_2(t) &= c_{42}\tilde{f}_4(x_4(t)) + c_{52}\tilde{f}_5(x_5(t)) - \tilde{f}_2(x_2(t)),\\
\dot{x}_3(t) &= c_{43}\tilde{f}_4(x_4(t)) + c_{53}\tilde{f}_5(x_5(t)) - \tilde{f}_3(x_3(t)),\\
\dot{x}_4(t) &= c_{64}\tilde{f}_6(x_6(t)) - \tilde{f}_4(x_4(t)),\\
\dot{x}_5(t) &= c_{65}\tilde{f}_6(x_6(t)) - \tilde{f}_5(x_5(t)),\\
\dot{x}_6(t) &= d(t) + c_{16}\tilde{f}_1(x_1(t)) - \tilde{f}_6(x_6(t)),
\end{aligned} \tag{1}$$

where the constants $c_{ji} \in \mathbb{R}_+$ can be interpreted as the number of orders of components from subsystem j to subsystem i.

By definition of $f_i(x,d) := \dot{x}_i(t), \; i = 1, \ldots, 6$, $x := (x_1, \ldots, x_6)^T$ and $f(x,u) := (f_1(x,d), \ldots, f_6(x,d))^T$ we can write the whole system as

$$\dot{x}(t) = f(x(t), d(t)), \; t \in \mathbb{R}_+. \tag{2}$$

Now the question arises, under which conditions the subsystems are stable, which means that the states of all subsystems will not increase to infinity. In other words, under which conditions all states of the subsystems and therefor the whole system are bounded, which means stability of the production network ?

3 Mathematical background

For the investigation of the stability of system (1) and (2) respectively we will need some mathematical results. We present a stability property and a tool how to check, if the system has the stability property.

We consider nonlinear dynamical systems of the form

$$\dot{x}(t) = f(x(t), u(t)), \tag{3}$$

where $t \in \mathbb{R}_+$ is the time, $\dot{x}(t)$ the derivate of the state $x(t) \in \mathbb{R}^N$ with initial value x_0, input $u(t) \in \mathbb{R}^m$, which is an essentially bounded measureable function and $f : \mathbb{R}^{N+m} \to \mathbb{R}^N$ nonlinear. To have existence and uniqueness of a solution of (3), function f has to be continuous and locally Lipschitz in x uniformly in u. The solution is denoted by $x(t; x_0, u)$ or $x(t)$ in short.

To describe the given production network we generalize (3) and consider $n \in \mathbb{N}$ interconnected systems. These are in general nonlinear dynamical systems of the form

$$\dot{x}_i(t) = f_i\left(x_1(t), \ldots, x_n(t), u_i(t)\right),\ i = 1, \ldots, n \tag{4}$$

where $t \in \mathbb{R}_+$; $x_i(t) \in \mathbb{R}^{N_i}$, $u_i(t) \in \mathbb{R}^{M_i}$, which are essentially bounded measureable functions, $f_i : \mathbb{R}^{\sum_j N_j + M_i} \to \mathbb{R}^{N_i}$, $i = 1, \ldots, n$, where f_i has to be continuous and locally Lipschitz in $x = \left(x_1^T, \ldots, x_n^T\right)^T$ uniformly in u_i. We consider x_j as input and u_i as external input of the i-th subsystem $i, j = 1, \ldots, n$ $i \neq j$. The solution is denoted by $x(t; x_i^0, x_j : j \neq i, u_i)$ or $x(t)$ in short.

If we define $N := \sum_{i=1}^n N_i$, $m := \sum_{i=1}^n M_i$, $x := (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$ and $f := (f_1^T, \ldots, f_n^T)^T$, then (4) becomes

$$\dot{x}(t) = f(x(t), u(t)),\ t \in \mathbb{R}_+. \tag{5}$$

We denote the standard euclidian norm in $\mathbb{R}^n$ by $\|\cdot\|$ and the essential supremum norm on essentially bounded functions u in $\mathbb{R}_+$ by $\|u\|_\infty$. We will need some classes of functions to define the stability property, which we want to use. A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is said to be *positive definite*, if $f(0) = 0$ and $f(x) > 0$, $\forall x \in \mathbb{R}^n$ holds. A class $\mathcal{K}$ function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is continuous, $\gamma(0) = 0$ and strictly increasing. If it is additionally unbounded then it is of class $\mathcal{K}_\infty$. We call a function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ of class $\mathcal{KL}$ if β is continuous, $\beta(\cdot, t) \in \mathcal{K}$ and $\beta(r, \cdot)$ strictly decreasing with $\lim_{t \to \infty} \beta(r, t) = 0$, $\forall t, r \geq 0$.

Now we define LISS and ISS respectively for each subsystem of (4). For system (3) the definition of LISS and ISS respectively can be found for example in [3] and [7] respectively.

Definition 1. *The i-th subsystem of (4) is called LISS*, if there exist constants $\rho_i > 0$, $\rho_i^u > 0$, γ_{ij}, $\gamma_i \in \mathcal{K}_\infty$ and $\beta_i \in \mathcal{KL}$, such that for all initial values $\left\|x_i^0\right\| \leq \rho_i$ and all inputs $\|u_i\|_\infty \leq \rho_i^u$ the inequality

$$\left\|x_i\left(t,x_i^0,x_j: j\neq i,u_i\right)\right\| \leq \max\left\{\beta_i\left(\left\|x_i^0\right\|,t\right), \max_{j\neq i}\gamma_{ij}\left(\left\|x_j\right\|_\infty\right), \gamma_i\left(\|u_i\|_\infty\right)\right\} \quad (6)$$

is satisfied $\forall\, t \in \mathbb{R}_+$. γ_{ij} and γ_i are called (nonlinear) gains.

Note that, if $\rho_i, \rho_i^u = \infty$ then the i-th subsystem is (global) ISS (see [1]). LISS and ISS respectively means that the norm of the trajectories of each subsystem is bounded.

Furthermore we define the *gain matrix* $\Gamma := (\gamma_{ij}),\ i,j = 1,\dots,n,\ \gamma_{ii} = 0$, which defines a map $\Gamma:\ \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\Gamma(s) := \left(\max_j \gamma_{1j}(s_j), \dots, \max_j \gamma_{nj}(s_j)\right)^T,\ s \in \mathbb{R}_+^n. \quad (7)$$

Previous investigations of two interconnected systems established a small gain condition (see [5] and [6]). In [1] an ISS small gain theorem for general networks was proved, where the small gain condition is of the form

$$\Gamma(s) \not\geq s,\ \forall\, s \in \mathbb{R}_+^n \setminus \{0\}. \quad (8)$$

Notation $\not\geq$ means that there is at least one component $i \in \{1,\dots,n\}$ such that $\Gamma(s)_i < s_i$. Here we note a local version of the small gain condition.

Definition 2. Γ satisfies the *local small gain condition (LSGC) on* $[0,w^*]$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\geq s,\ \forall s \in [0,w^*],\ s \neq 0. \quad (9)$$

Further informations of (9) can be found in [3]. We note the local version of the small gain theorem:

Theorem 1. *All subsystems of (4) satisfy (6). Suppose Γ satisfies LSGC. Then there exist constants $\rho,\ \rho^u > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that the whole system (5) is LISS.*

The proof can be found in [3], Theorem 4.2.

An important tool to verify LISS and ISS respectively of a system of the form (4) are Lyapunov functions. For systems of the form (3) one can find the definition of Lyapunov functions for example in [6] and [3].

Definition 3. A smooth function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$ is called *LISS Lyapunov function of the i-th subsystem of system (4)*, if it satisfies the following two conditions:
There exist functions $\psi_{1i},\ \psi_{2i} \in \mathcal{K}_\infty$ such that

$$\psi_{1i}(\|x_i\|) \leq V_i(x_i) \leq \psi_{2i}(\|x_i\|),\ \forall\, x_i \in \mathbb{R}^{N_i} \quad (10)$$

and there exists $\chi_{ij},\ \chi_i \in \mathcal{K}_\infty$, a positive function μ_i and constants $\rho_i,\ \rho_i^u > 0$ such

$$V_i(x_i) \geq \max\left\{\max_j \chi_{ij}\left(V_j(x_j)\right), \chi_i\left(\|u_i\|\right)\right\} \ \Rightarrow\ \nabla V_i(x_i)\cdot f_i(x,u) \leq -\mu_i\left(V_i(x_i)\right) \quad (11)$$

for all $x_i \in \mathbb{R}^{N_i}$, $\left\|x_i^0\right\| \le \rho_i$, $u_i \in^{M_i}$, $\|u_i\|_\infty \le \rho_i^u$, $\chi_{ii} = 0$. Functions χ_{ij} are called *LISS Lyapunov gains.*

Note that, if $\rho_i, \rho_i^u = \infty$ then the LISS Lyapunov function of the i-th subsystem becomes an ISS Lyapunov function of the i-th subsystem (see [4]).

Condition (10) implies that V_i is proper, positive definite and bounded from below and above. The second condition (11) of a Lyapunov function is important and means that if $V_i(x_i) \ge \max\left\{\max_j \chi_{ij}\left(V_j(x_j)\right), \chi_i\left(\|u_i\|\right)\right\}$ holds true, then the total derivate of V_i along the trajectories is negative. This can be interpreted as the trajectory is bounded from above. As long as $V_i(x_i) \ge \max\left\{\max_j \chi_{ij}\left(V_j(x_j)\right), \chi_i\left(\|u_i\|\right)\right\}$ is not true we cannot say anything about the behavior of the trajectory.

To check if a the whole system of the form (5) has the LISS or ISS property one has to find a LISS Lyapunov function or ISS Lyapunov function respectively for each subsystem of the form (4). If there exists a LISS or ISS Lyapunov function for the subsystem then it has the LISS or ISS property respectively. Furthermore, if the LISS Lyapunov gains or ISS Lyapunov gains satisfy the small gain condition, then the whole system of the form (5) is LISS or ISS respectively (see [2], [3] or [4]).

With this mathematical theory we are able to derive conditions for which the subsystems and the whole system are stable. This will be presented in the next section.

4 Stability of the model

In this section we investigate all six subsystems of (1) to check if they have the LISS or ISS property respectively. Therefor we will choose a Lyapunov function candidate for each subsystem and check if conditions (10) and (11) are satisfied.

Remark 1. All subsystems of (1) are nonnegative, since for $x_i = 0$ and no input the term $f_i(x_i)$ is zero, $i = 1, \ldots, n$.

We choose the Lyapunov function candidate $V_i(x_i) = x_i$ for all six subsystems. V_i satisfies condition (10).

For the investigation of the first subsystem we define

$$\chi_{j1}(x_j) := -\ln\left(1 - \frac{c_{21}\alpha_2 + c_{31}\alpha_3}{(1-\varepsilon_{j1})\alpha_1}\left(1 - \exp(-x_j)\right)\right) \le x_1 = V_1(x_1),$$

$j = 2, 3,\ 1 > \varepsilon_{j1} > 0$, which implies

$$c_{j1}\alpha_j\left(1 - \exp(-x_j)\right) \le \frac{c_{j1}\alpha_j}{c_{21}\alpha_2 + c_{31}\alpha_3}(1-\varepsilon_{j1})\alpha_1\left(1 - \exp(-x_1)\right).$$

To guarantee that χ_{j1} is proper we claim

$$c_{21}\alpha_2 + c_{31}\alpha_3 < \alpha_1(1-\varepsilon_{j1}) < \alpha_1. \tag{12}$$

With this considerations it follows

$$
\begin{aligned}
&\nabla V_1(x_1(t))\,(f_1(x_1(t),\dots,x_6(t),d(t))\\
&= c_{21}\alpha_2(1-\exp(-x_2))+c_{31}\alpha_3(1-\exp(-x_3))-\alpha_1(1-\exp(-x_1))\\
&\leq \left(\frac{(1-\varepsilon_{21})\alpha_1 c_{21}\alpha_2}{c_{21}\alpha_2+c_{31}\alpha_3}+\frac{(1-\varepsilon_{31})\alpha_1 c_{31}\alpha_3}{c_{21}\alpha_2+c_{31}\alpha_3}-\alpha_1\right)(1-\exp(-x_1))\\
&\leq -\varepsilon_1\alpha_1(1-\exp(-x_1)) = -\mu_1(V_1(x_1(t)))
\end{aligned}
$$

where $\varepsilon_1 := \min\{\varepsilon_{21},\varepsilon_{31}\}$ and $\mu_1(r) := \varepsilon_1\alpha_1\,(1-\exp(-r))$ is a positive definite function.

For simplication one can choose $\varepsilon_{j1},\ j=2,3$ close to 0. The reason of the introduction of the constant value ε_{j1} is to guarantee that μ_1 is positive definite. V_1 satisfies condition (11) and is the ISS Lyapunov function of the first subsystem from which we know that the first subsystem has the ISS property for all $x_j\in\mathbb{R}_+,\ j=1,2,3$, if condition (12) holds.

For subsystems two to five we do similar calculations and get the gains

$$
\begin{aligned}
\chi_{j2}(x_j) &:= -\ln\left(1-\frac{c_{42}\alpha_4+c_{52}\alpha_5}{(1-\varepsilon_{j2})\alpha_2}\,(1-\exp(-x_j))\right),\ 1>\varepsilon_{j2}>0,\ j=4,5,\\
\chi_{j3}(x_j) &:= -\ln\left(1-\frac{c_{43}\alpha_4+c_{53}\alpha_5}{(1-\varepsilon_{j3})\alpha_3}\,(1-\exp(-x_j))\right),\ 1>\varepsilon_{j3}>0,\ j=4,5,\\
\chi_{6j}(x_6) &:= -\ln\left(1-\frac{c_{6j}\alpha_6}{(1-\varepsilon_{6j})\alpha_j}\,(1-\exp(-x_6))\right),\ 1>\varepsilon_{6j}>0,\ j=4,5
\end{aligned}
$$

and conditions

$$
\alpha_2 > c_{42}\alpha_4+c_{52}\alpha_5,\ \alpha_3 > c_{43}\alpha_4+c_{53}\alpha_5,\ \alpha_4 > c_{64}\alpha_6,\ \alpha_5 > c_{65}\alpha_6 \tag{13}
$$

for which the subsystems two to five have the ISS property.

For the sixth subsystem from

$$
\begin{aligned}
\chi_6\,(d(t)) &:= -\ln\left(1-\frac{d(t)(\|d\|_\infty+c_{16}\alpha_1)}{\|d\|_\infty\,(1-\varepsilon_{d6})\alpha_6}\right)\leq x_6 = V_6(x_6), \qquad (14)\\
\chi_{16}(x_1) &:= -\ln\left(1-\frac{\|d\|_\infty+c_{16}\alpha_1}{(1-\varepsilon_{16})\alpha_6}(1-\exp(-x_1))\right)\leq x_6 = V_6(x_6),
\end{aligned}
$$

with $0<\varepsilon_{16},\varepsilon_{d6}<1$ we get

$$
\begin{aligned}
&\nabla V_6(x_6(t))f_6(x_1(t),\dots,x_6(t),d(t))\\
&= d(t)-\alpha_6\,(1-\exp(-x_6(t)))+c_{16}\alpha_1\,(1-\exp(-x_1(t)))\\
&\leq -\varepsilon_6\alpha_6\,(1-\exp(-x_6(t))) = \mu_6(V_6(x_6(t))),
\end{aligned}
$$

where $\varepsilon_6 := \min\{\varepsilon_{16},\varepsilon_{d6}\}$ and $\mu_6(r) := \varepsilon_6\alpha_6\,(1-\exp(-r))$ is positive definite, because to guarantee that χ_6 and χ_{16} are proper we claim

$$
\alpha_6 > \|d\|_\infty + c_{16}\alpha_1. \tag{15}
$$

Function χ_6 as defined in (14) is $\in \mathscr{K}$, but we can find a continuation of χ_6 such that the composed function is $\mathscr{K}_\infty$.

Hence V_6 satisfies condition (11) and we know that subsystem six has the LISS property for all $x_6^0 \in \mathbb{R}_+$ and $\|d\|_\infty < \alpha_6 - c_{16}\alpha_1 =: \rho^u$.

The gain matrix is of the form

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \chi_{16} \\ \chi_{21} & 0 & 0 & 0 & 0 & 0 \\ \chi_{31} & 0 & 0 & 0 & 0 & 0 \\ 0 & \chi_{42} & \chi_{43} & 0 & 0 & 0 \\ 0 & \chi_{52} & \chi_{53} & 0 & 0 & 0 \\ 0 & 0 & 0 & \chi_{64} & \chi_{65} & 0 \end{pmatrix}.$$

With $\exp(-r) < 1,\ r > 0 \ \Leftrightarrow\ (1-a)\exp(-r) < (1-a),\ 0 < a < 1 \ \Leftrightarrow\ \exp(-r) < 1 - a + a\exp(-r) \ \Leftrightarrow\ -\ln(1-a+a\exp(-r)) < r$ it follows

$$\begin{aligned} &\chi_{16}\circ\chi_{64}\circ\chi_{42}\circ\chi_{21}(r) \\ &= -\ln\left(1 - \frac{\|d\|_\infty + c_{16}\alpha_1}{(1-\varepsilon_{16})\alpha_6}\frac{c_{64}\alpha_6}{(1-\varepsilon_{64})\alpha_4}\frac{c_{42}\alpha_4 + c_{52}\alpha_5}{(1-\varepsilon_{42})\alpha_2}\frac{c_{21}\alpha_2 + c_{31}\alpha_3}{(1-\varepsilon_{21})\alpha_1}(1-\exp(-r))\right) \\ &< r,\ r > 0. \end{aligned}$$

By similar calculations is holds

$$\begin{aligned} &\chi_{16}\circ\chi_{64}\circ\chi_{42}\circ\chi_{21}(r) < r,\ \ \chi_{16}\circ\chi_{64}\circ\chi_{43}\circ\chi_{31}(r) < r, \\ &\chi_{16}\circ\chi_{65}\circ\chi_{52}\circ\chi_{21}(r) < r,\ \ \chi_{16}\circ\chi_{65}\circ\chi_{53}\circ\chi_{31}(r) < r, \end{aligned} \tag{16}$$

for $r > 0$. (16) is equivalent to the compliance of the small gain condition, if conditions (12), (13) and (15) are satisfied.

We conclude that all subsystems are LISS or ISS respectively and we can apply Theorem 1 such that the whole system is LISS for all $x_0 \in \mathbb{R}_+^6$ and $\|d\|_\infty < \rho^u$ with additional conditions (12), (13) and (15).

In the next section we make simulations to illustrate the derived results of this section.

5 Simulation results

To demonstrate the results of the previous section, we simulate all subsystems with Matlab.

At first we will choose values for the parameters c_{ji}:

$$c_{16} = \frac{1}{10000},\ c_{21} = 4,\ c_{31} = 3,\ c_{42} = 4\,c_{43} = 9,\ c_{52} = 6,\ c_{53} = 2,\ c_{64} = 8,\ c_{65} = 4.$$

Consider constant orders $d \equiv 20$. Then conditions (12), (13) and (15) become

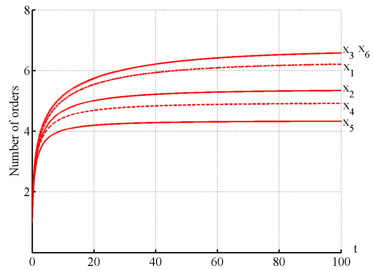

Fig. 2 Number of orders, if stability conditions are satisfied

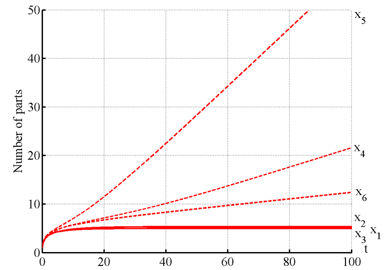

Fig. 3 Number of orders, if stability conditions are not satisfied

$$\alpha_1 > 4\alpha_2 + 3\alpha_3,\ \alpha_2 > 4\alpha_4 + 6\alpha_5,\ \alpha_3 > 9\alpha_4 + 2\alpha_5,$$
$$\alpha_4 > 8\alpha_6,\ \alpha_5 > 4\alpha_6,\ \alpha_6 > 20 + 0.0001\alpha_1.$$

By solving this system of linear inequalities we get the condition

$$\alpha := (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)^T > (9731.55,\ 1174.5,\ 1677.86,\ 167.79,\ 83.9,\ 20.98)^T$$

With the choice $\alpha = (9750,\ 1180,\ 1680,\ 169,\ 85,\ 21)^T$ and $x_0 = (1,1,1,1,1,1)^T$ the simulation results are presented in Figure 2, where the number of orders of each subsystem for time t are displayed. We can see, that all subsystems are bounded.

Now we choose the maximum production rates only a bit smaller:

$$\alpha = (9730,\ 1174,\ 1677,\ 167,\ 83,\ 20.9)^T$$

The simulation results are displayed in Figure 3. We can see that the trajectories of the subsystems one to three are bounded, but the trajectories of the subsystems four to six are unbounded, which means that the whole system is not stable.

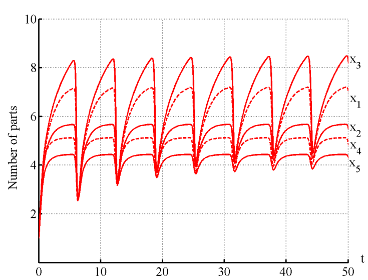

Fig. 4 Simulation results for x_1 to x_5 with $d(t) = 20 \cdot (\sin(t) + 1)$

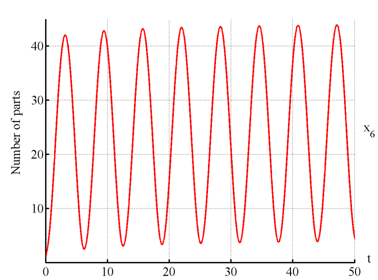

Fig. 5 Simulation results for x_6 with $d(t) = 20 \cdot (\sin(t) + 1)$

By further simulations of the system we discover that for other inputs where $\|d\|_\infty < \rho^u$ is not satisfied, the system is also stable.

We consider all values c_{ji} as before, choose the maximum production rates $\alpha = (9750,\ 1180,\ 1680,\ 169,\ 85,\ 21)^T$ such that conditions (12) and (13) are satisfied and replace d by $d(t) = 20 \cdot (\sin(t) + 1)$. So it is $\|d\|_\infty = 40 > \rho^u$, but by simulation results, which are presented in Figures 4 and 5, all subsystems and therefor the whole system are stable.

By mathematical theory used in this paper it is not possible to cover all inputs for which the system is stable, in particular oscillating inputs. This is an actual mathematical problem to find the domain of stability as large as possible.

6 Conclusions and outline

In this paper we have described a model for networks of autonomous production plants. This model was investigated on stability. In particular necessary condition for its stable behavior were provided. This paper illustrates a general approach for modelling and analysis of autonomous logistic systems.

For validation of the provided methods a comparison of the obtained results with simulations provided by discrete event simulation is of interest and is planned for the future research.

Acknowledgements This research is funded by the German Research Foundation (DFG) as part of the Collaborative Research Centre 637 "Autonomous Cooperating Logistic Processes: A Paradigm Shift and its Limitations" (SFB 637).

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