Exponential Synchronization of Master-Slave Neural Networks with Time-Delays

Hamid Reza Karimi, Sergey Dashkovskiy

Abstract— **This paper establishes an exponential H_∞ synchronization method for a class of master and slave neural networks (MSNNs) with mixed time-delays, where the delays comprise different neutral, discrete and distributed time-delays and the class covers the Lipschitz-type nonlinearity case. By introducing a novel discretized Lyapunov-Krasovskii functional in order to minimize the conservatism in the stability problem of the system and also using some free weighting matrices, new delay-dependent sufficient conditions are derived for designing a delayed state-feedback control as a synchronization law in terms of linear matrix inequalities (LMIs). The controller guarantees the exponential H_∞ synchronization of the two coupled MSNNs regardless of their initial states. Detailed comparisons with different number of segments are made and numerical simulations are carried out to demonstrate the effectiveness of the established synchronization laws.**

I. Introduction

In the last few years, synchronization in neural networks has received a great deal of interest among scientists from various fields [1, 2]. The first idea of synchronizing two identical chaotic systems with different initial conditions was introduced by Pecora and Carroll [3], and the method was realized in electronic circuits. The methods for synchronization of the chaotic systems have been widely studied in recent years, and many different methods have been applied theoretically and experimentally to synchronize chaotic systems, such as feedback control [4-10], adaptive control [11-15], backstepping [16] and sliding mode control [17-21]. Recently, the theory of incremental input-to-state stability to the problem of synchronization in a complex dynamical network of identical nodes was studied in [22].

On the other hand, in practice, due to the finite switching speed of amplifiers or finite speed of information processing, time delays are often encountered in hardware implementation [23-27], which may be a source of oscillation, divergence, and instability in neural networks. Therefore, the stability problems of neural networks with time delay have gained great research interest. Recently, both delay-independent and delay-dependent sufficient conditions have been proposed to verify the asymptotical or exponential stability of delay neural networks, see for instance the references [28-38] and references therein. Among these results, Mou et al. [39], Gau et al. [40], He et al. [41], and Park [42] very recently derived an improved exponential stability condition over the existing criterion. It can be realized that, up to now, few attempts have been made towards solving the synchronization problem of neural networks with different neutral, discrete and distributed time-delays.

In this paper, we contribute to the further development of an exponential H_∞ synchronization method for a class of master and slave neural networks (MSNNs) with mixed time-delays, where the mixed delays comprise different neutral, discrete and distributed time-delays and the class covers the Lipschitz-type nonlinearity case. The main merit of the proposed method lies in the fact that it provides a convex problem via introduction of a complete quadratic Lyapunov-Krasovskii functional with additional decision variables such that the control gains can be found from the linear matrix inequality (LMI) formulations. Therefore, an appropriate discretized Lyapunov-Krasovskii functional is constructed in order to establish some less conservative delay-dependent sufficient conditions for designing a delayed state-feedback control as a synchronization law in terms of LMIs using some free weighting matrices. Then, the controller is developed based on state measurements so as to guarantee that the exponential H_∞ synchronization of the two coupled MSNNs is achieved regardless of their initial states. All the developed results are tested on a representative example.

Notations. The superscript 'T' stands for matrix transposition; $\Re^n$ denotes the n-dimensional Euclidean space; $\Re^{n\times m}$ is the set of all real m by n matrices. The vector v_i denotes the unit column vector having a '1' element on its ith row and zeros elsewhere. $\|.\|$ refers to the Euclidean vector norm or the induced matrix 2-norm and $diag\{\cdots\}$ represents a block diagonal matrix. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalue of the square matrix A. The operator $sym\{A\}$ denotes $A+A^T$. The symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix.

II. Problem description

In this paper, the master-slave neural networks (MSNNs) with mixed time-delays are described as follows:

H.R. Karimi is with the Faculty of Technology and Science, University of Agder, Serviceboks 509, N-4898 Grimstad, Norway (e-mail: hamid.r.karimi@uia.no).

S. Dashkovskiy is with the Centre for Industrial Mathematics of the University of Bremen, Bibliothekstr. 1, 28359 Bremen, Germany.

ISBN 978-963-311-369-1

$$\begin{cases} \dot{x}(t) = -A\,x(t) + W_1\, f(x(t)) + W_2\, g(x(t-\tau_1)) + W_3\, \dot{x}(t-\tau_2) \\ \qquad + W_4 \int_{t-\tau_3}^{t} h(x(s))\, ds + o, \\ x(t) = \phi(t), \qquad t \in [-\bar{\tau}, 0], \\ z_x(t) = C_1\, x(t) + C_2\, x(t-\tau_1) + C_3 \int_{t-\tau_3}^{t} h(x(s))\, ds, \end{cases} \tag{1}$$

$$\begin{cases} \dot{y}(t) = -A\,y(t) + W_1\, f(y(t)) + W_2\, g(y(t-\tau_1)) + W_3\, \dot{y}(t-\tau_2) \\ \qquad + W_4 \int_{t-\tau_3}^{t} h(y(s))\, ds + E\,w(t) + u(t) + o, \\ y(t) = \varphi(t), \qquad t \in [-\bar{\tau}, 0], \\ z_y(t) = C_1\, y(t) + C_2\, y(t-\tau_1) + C_3 \int_{t-\tau_3}^{t} h(y(s))\, ds, \end{cases} \tag{2}$$

with $x(t) = [x_1(t), x_2(t), \cdots, x_n(t)]^T \in \Re^n$, $y(t) = [y_1(t), y_2(t), \cdots, y_n(t)]^T \in \Re^n$ where $x_i(t)$ and $y_i(t)$ are the master system's state vector and the slave system's state vector associated with the ith neuron; $u(t) \in \Re^n$ is a coupled term which is considered as the control input; $w(t) \in \Re^q$ is the disturbance and $z_x(t),\ z_y(t) \in \Re^s$ are the controlled outputs of the master and slave systems, respectively. $f(x(t)) = [f_1(x_1(t)), f_2(x_2(t)), \cdots, f_n(x_n(t))]^T$, $g(x(t-\tau_1)) = [g_1(x_1(t-\tau_1)), g_2(x_2(t-\tau_1)), \cdots, g_n(x_n(t-\tau_1))]^T$ and $h(x(t)) = [h_1(x_1(t)), h_2(x_2(t)), \cdots, h_n(x_n(t))]^T$ denote the activation functions, $A = diag\{a_i\} > 0$, the vector $o = [o_1, o_2, \cdots, o_n]^T$ is the constant external input and the scalars $\tau_i > 0$, for $i = 1,2,3$, denote the known neutral, discrete and distributed time-delays, respectively, and $\bar{\tau} := \max\{\tau_1, \tau_2, \tau_3\}$. The initial functions $\phi(t)$ and $\varphi(t)$ are continuously differentiable functionals.

Definition 1. ([37]) The MSNNs (1)-(2) are synchronized globally exponentially if there exist scalars $\alpha > 0$ and $M \geq 1$ such that $|e(t)| \leq M\ e^{-\alpha t}[\|\zeta\| + \|\dot{\zeta}\|]$, where $\zeta(t) \in C([-\bar{\tau}, 0]; \Re^n)$ is an initial condition and $e(t) = x(t) - y(t)$ is the synchronization error such α and M are called the exponential decay rate and decay coefficient, respectively.

Let $\hat{e}(t) = e^{\alpha t} e(t)$. The error dynamics between (1)-(2), namely synchronization error network, can be expressed by

$$\begin{cases} \dot{\hat{e}}(t) = -(A - \alpha I)\hat{e}(t) + W_1\, \hat{\psi}_1(\hat{e}(t)) + W_2\, e^{\alpha\tau_1}\, \hat{\psi}_2(\hat{e}(t-\tau_1)) - \alpha e^{\alpha\tau_2}\, W_3\, \hat{e}(t-\tau_2) \\ \qquad + e^{\alpha\tau_2}\, W_3\, \dot{\hat{e}}(t-\tau_2) + W_4 \int_{t-\tau_3}^{t} e^{\alpha(t-s)}\, \hat{\psi}_3(\hat{e}(s))\, ds - E\,\hat{w}(t) - \hat{u}(t), \\ \hat{e}(t) = \phi_e(t;\alpha), \qquad t \in [-\bar{\tau}, 0], \\ \hat{z}_e(t) = C_1\, \hat{e}(t) + C_2 e^{\alpha\tau_1}\, \hat{e}(t-\tau_1) + C_3 \int_{t-\tau_3}^{t} e^{\alpha(t-s)}\, \hat{\psi}_3(\hat{e}(s))\, ds, \end{cases} \tag{3}$$

where $\hat{u}(t) = e^{\alpha t} u(t)$, $\hat{w}(t) = e^{\alpha t} w(t)$, $\hat{z}_e(t) = \hat{z}_x(t) - \hat{z}_y(t) = e^{\alpha t}(z_x(t) - z_y(t))$, $\hat{\psi}_1(\hat{e}(t)) = [\hat{\psi}_{11}(\hat{e}_1(t)), \hat{\psi}_{12}(\hat{e}_2(t)), \cdots, \hat{\psi}_{1n}(\hat{e}_n(t))]^T$, $\hat{\psi}_2(\hat{e}(t-\tau_1)) = [\hat{\psi}_{21}(\hat{e}_1(t-\tau_1)), \hat{\psi}_{22}(\hat{e}_2(t-\tau_1)), \cdots, \hat{\psi}_{2n}(\hat{e}_n(t-\tau_1))]^T$ and $\hat{\psi}_3(\hat{e}(t))[\hat{\psi}_{31}(\hat{e}_1(t)), \hat{\psi}_{32}(\hat{e}_2(t)), \cdots, \hat{\psi}_{3n}(\hat{e}_n(t))]^T$ with $\hat{\psi}_{1i}(\hat{e}_i(t)) = e^{\alpha t}(f_i(x_i(t)) - f_i(y_i(t)))$, $\hat{\psi}_{2i}(\hat{e}_i(t)) = e^{\alpha t}(g_i(x_i(t)) - g_i(y_i(t)))$, $\hat{\psi}_{3i}(\hat{e}_i(t)) = e^{\alpha t}(h_i(x_i(t)) - h_i(y_i(t)))$ and $\phi_e(t;\alpha) = e^{\alpha t}(\phi(t) - \varphi(t))$.

In this paper, the following assumptions are needed:

A1) ([37]) The nonlinear functions $f_i(s), g_i(s), h_i(s)$, for any $i = 1, \cdots, n$, satisfy, respectively,

$$f_i^- \leq \frac{f_i(s_1) - f_i(s_2)}{s_1 - s_2} \leq f_i^+,\ g_i^- \leq \frac{g_i(s_1) - g_i(s_2)}{s_1 - s_2} \leq g_i^+,\ h_i^- \leq \frac{h_i(s_1) - h_i(s_2)}{s_1 - s_2} \leq h_i^+,$$

for $\forall s_1, s_2 \in \Re$ where $f_i^-, f_i^+, g_i^-, g_i^+, h_i^-, h_i^+$ are constants.

A2) Let the difference operator $D: C([-\max\{\tau_1, \tau_2, \tau_3\}, 0], \Re^n) \to \Re^n$ given by $Dx_t = x(t) - W_3\, x(t-\tau_2)$ be delay-independently stable with respect to all delays [10]. A sufficient condition for A2) is that all the eigenvalues of the the matrix W_3 are inside the unit circle, i.e. $\lambda_{\max}(W_3) < 1$.

Remark 1. One can easily check that, for any $i = 1, \cdots, n$, the functions $\hat{\psi}_{1i}(\hat{e}_i(t)), \hat{\psi}_{2i}(\hat{e}_i(t))$ and $\hat{\psi}_{3i}(\hat{e}_i(t))$ satisfy, respectively:

$$f_i^- \leq \frac{\hat{\psi}_{1i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq f_i^+,\ g_i^- \leq \frac{\hat{\psi}_{2i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq g_i^+,\ h_i^- \leq \frac{\hat{\psi}_{3i}(\hat{e}_i(t))}{\hat{e}_i(t)} \leq h_i^+.$$

The problem to be addressed in this paper is as follows: given the delayed MSNNs (1)-(2) with a prescribed level of disturbance attenuation $\gamma > 0$, find a driving signal $\hat{u}(t)$,

$$\hat{u}(t) = K_1\, \hat{e}(t) + K_2\, \hat{e}(t-\tau_1) + K_3 \int_{t-\tau_3}^{t} \hat{e}(s)\, ds \tag{4}$$

where $\{K_i\}_{i=1}^3$ are the control gains to be determined such that

1) the network (3) is globally exponentially stable;
2) under zero initial conditions and for all non-zero $w(t) \in L_2[0, \infty]$, the H_∞ performance measure, i.e., $J_\infty = \int_0^\infty [\hat{z}_e^T(t)\, \hat{z}_e(t) - \gamma^2 \hat{w}^T(t)\, \hat{w}(t)]\ dt$, satisfies $J_\infty < 0$;

in this case, the MSNNs (1)-(2) are said to be asymptotically stable with an H_∞ performance measure.

III. Main Results

In this section, sufficient conditions for the solvability of the problem of the delayed state-feedback control design are proposed using the Lyapunov method and an LMI approach.

Theorem 1. Let $h_i = \tau_i / N$ be given for any positive integer N. For the MSNNs (1)-(2), a state feedback controller given in the form (4) exists such that the synchronization error network (3) is globally exponentially stable with an exponential synchronization degree $\alpha > 0$ and a disturbance attenuation level $\gamma > 0$, if there exist some scalars $\delta, d_l, \sigma_l, \rho_l, \lambda_l$ ($l = 1,2,\cdots,N$), matrices $P_2, L_1, L_2, L_3,\ Q_i,\ S_i, H_i, R_{i,j} = R_{i,j}^T, T_{i,j} = T_{i,j}^T$ ($i, j = 0,1,\cdots,N$) and positive-definite matrices $P_1, Z_1, Z_2, \overline{U}_1, \overline{U}_2$ satisfying the following LMIs

$$\begin{bmatrix} P_1 & \tilde{Q} \\ * & \tilde{R} + \tilde{S} \end{bmatrix} > 0,\ \begin{bmatrix} \overline{U}_1 & -\overline{U}_1 \\ * & S_d \end{bmatrix} > 0,\ \begin{bmatrix} \overline{U}_2 & -\overline{U}_2 \\ * & H_d \end{bmatrix} > 0, \tag{5a-c}$$

$$\Pi := \begin{bmatrix} \hat{\Xi}_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3\overline{U}_1 & 0 \\ * & * & * & * & -3\overline{U}_2 \end{bmatrix} < 0, \tag{5d}$$

where

$$\hat{\Xi}_e = \begin{bmatrix} \Sigma_{11} & \begin{bmatrix} -L_2 - Q_N + e^{\alpha\tau_1} C_1^T C_2 \\ -\delta L_2 \end{bmatrix} & \begin{bmatrix} -\alpha\, P_2^T W_3 e^{\alpha\tau_2} \\ -\alpha\delta\, P_2^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} P_2^T W_3 e^{\alpha\tau_2} \\ \delta\, P_2^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} -L_3 \\ -\delta L_3 \end{bmatrix} & \Sigma_{15} \\ * & -S_N + e^{2\alpha\tau_1} C_2^T C_2 & 0 & 0 & 0 & 0 \\ * & * & -H_N & 0 & 0 & 0 \\ * & * & * & -U_1 & 0 & 0 \\ * & * & * & * & -U_2 & 0 \\ * & * & * & * & * & \Sigma_{55} \end{bmatrix}$$

$$\Sigma_{11} = sym(\begin{bmatrix} -P_2^T(A-\alpha I) - L_1 & P_1 - P_2^T \\ -\delta P_2^T (A-\alpha I) - \delta L_1 & -\delta P_2^T \end{bmatrix}) + diag\{sym(Q_0) + S_0 + H_0$$
$$+ \tau_3^2 U_2 - F^+\Lambda_1 F^- - G^+\Lambda_2 G^- - H^+\Lambda_3 H^-, U_1\}$$

$$\Sigma_{15} = \begin{bmatrix} \begin{bmatrix} P_2^T W_1 + \frac{1}{2}(F^+ + F^-)\Lambda_1 \\ \delta P_2^T W_1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(G^+ + G^-)\Lambda_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(H^+ + H^-)\Lambda_3 \\ 0 \end{bmatrix} \\ 0 & 0 & 0 \end{bmatrix}$$
$$\begin{bmatrix} \begin{bmatrix} P_2^T W_4 + C_1^T C_3 \\ \delta P_2^T W_4 \end{bmatrix} & \begin{bmatrix} P_2^T W_2 e^{\alpha\tau_1} \\ \delta P_2^T W_2 e^{\alpha\tau_1} \end{bmatrix} & \begin{bmatrix} -P_2^T E \\ -\delta P_2^T E \end{bmatrix} \\ 0 & 0 & 0 \end{bmatrix}$$

$$\Sigma_{55} = diag\{-\Lambda_1, Z_1 - \Lambda_2, \tau_3^2 Z_2 - \Lambda_3, -e^{-2\alpha\tau_3} Z_2 + C_3^T C_3, -Z_1, -\gamma^2 I\},$$

$$R_{ds} = h_1 \begin{bmatrix} R_{0,0} - R_{1,1} & R_{0,1} - R_{1,2} & \cdots & R_{0,N-1} - R_{1,N} \\ R_{1,0} - R_{2,1} & R_{1,1} - R_{2,2} & \cdots & R_{1,N-1} - R_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N-1,0} - R_{N,1} & R_{N-1,1} - R_{N,2} & \cdots & R_{N-1,N-1} - R_{N,N} \end{bmatrix},$$

$$T_{ds} = h_2 \begin{bmatrix} T_{0,0} - T_{1,1} & T_{0,1} - T_{1,2} & \cdots & T_{0,N-1} - T_{1,N} \\ T_{1,0} - T_{2,1} & T_{1,1} - T_{2,2} & \cdots & T_{1,N-1} - T_{2,N} \\ \vdots & \vdots & \ddots & \vdots \\ T_{N-1,0} - T_{N,1} & T_{N-1,1} - T_{N,2} & \cdots & T_{N-1,N-1} - T_{N,N} \end{bmatrix},$$

$$D^s = h_1 \begin{bmatrix} 2Q_1^a + R_{0,1}^s & 2Q_2^a + R_{0,2}^s & \cdots & 2Q_N^a + R_{0,N}^s \\ Q_1^s & Q_2^s & \cdots & Q_N^s \\ -R_{N,1}^s & -R_{N,2}^s & \cdots & -R_{N,N}^s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$\widetilde{R} = \begin{bmatrix} R_{0,0} & R_{0,1} & \cdots & R_{0,N} \\ R_{1,0} & R_{1,1} & \cdots & R_{1,N} \\ \vdots & \vdots & \ddots & \vdots \\ R_{N,0} & R_{N,1} & \cdots & R_{N,N} \end{bmatrix}, O^s = h_2 \begin{bmatrix} T_{0,1}^s & T_{0,2}^s & \cdots & T_{0,N}^s \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,1}^s & T_{N,2}^s & \cdots & T_{N,N}^s \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

$$D^a = h_1 \begin{bmatrix} R_{0,1}^a & R_{0,2}^a & \cdots & R_{0,N}^a \\ Q_1^a & Q_2^a & \cdots & Q_N^a \\ -R_{N,0}^a & -R_{N,1}^a & \cdots & -R_{N,N-1}^a \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, O^a = h_2 \begin{bmatrix} T_{0,1}^a & T_{0,2}^a & \cdots & T_{0,N}^a \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ T_{N,0}^a & T_{N,1}^a & \cdots & T_{N,N-1}^a \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

with $S_d = diag\{S_0 - S_1, S_1 - S_2, \cdots, S_{N-1} - S_N\}$, $H_d = diag\{H_0 - H_1, H_1 - H_2, \cdots, H_{N-1} - H_N\}$, $\widetilde{Q} = [Q_0, Q_1, \cdots, Q_N]$, $\widetilde{S} = 1/h_1\ diag\{S_0, S_1, \cdots, S_N\}$, $Q_p^s = (Q_p + Q_{p-1})/2$, $Q_p^a = (Q_p - Q_{p-1})/2$, $R_{p,q}^s = (R_{p,q} + R_{p,q-1})/2$, $R_{p,q}^a = (R_{p,q} - R_{p,q-1})/2$, $T_{p,q}^s = (T_{p,q} + T_{p,q-1})/2$, $T_{p,q}^a = (T_{p,q} - T_{p,q-1})/2$, $F^+ = diag\{f_1^+, f_2^+, \cdots, f_N^+\}$, $G^+ = diag\{g_1^+, g_2^+, \cdots, g_N^+\}$, $H^+ = diag\{h_1^+, h_2^+, \cdots, h_N^+\}$, $F^- = diag\{f_1^-, f_2^-, \cdots, f_N^-\}$, $G^- = diag\{g_1^-, g_2^-, \cdots, g_N^-\}$, $H^- = diag\{h_1^-, h_2^-, \cdots, h_N^-\}$, $\Lambda_1 = diag\{\sigma_1, \sigma_2, \cdots, \sigma_N\}$, $\Lambda_2 = diag\{\rho_1, \rho_2, \cdots, \rho_N\}$ and $\Lambda_3 = diag\{\lambda_1, \lambda_2, \cdots, \lambda_N\}$. Then, the controller gains in (4) are given by $K_i = (P_2^T)^{-1} L_i$.

Proof: Choose a Lyapunov-Krasovskii functional candidate as

$$V(t) = \sum_{i=1}^{6} V_i(t), \tag{6}$$

where

$$V_1(t) = \hat{e}(t)^T P_1 \hat{e}(t) + 2\hat{e}(t)^T \int_{-\tau_1}^{0} Q(\xi)\hat{e}(t+\xi)\ d\xi,$$
$$V_2(t) = \int_{t-\tau_1}^{t} \hat{\psi}_2(\hat{e}(s))^T Z_1 \hat{\psi}_2(\hat{e}(s))\ ds + \int_{t-\tau_2}^{t} \dot{\hat{e}}(s)^T U_1 \dot{\hat{e}}(s)\ ds,$$
$$V_3(t) = \int_{-\tau_1}^{0} \hat{e}(t+\xi)^T S(\xi)\hat{e}(t+\xi)\ d\xi + \int_{-\tau_1}^{0}\int_{-\tau_1}^{0} \hat{e}(t+s)^T R(s,\xi)\hat{e}(t+\xi)\ ds\, d\xi,$$
$$V_4(t) = \int_{-\tau_2}^{0} \hat{e}(t+\xi)^T H(\xi)\hat{e}(t+\xi)\ d\xi + \int_{-\tau_2}^{0}\int_{-\tau_2}^{0} \hat{e}(t+s)^T T(s,\xi)\hat{e}(t+\xi)\ ds\, d\xi,$$
$$V_5(t) = \int_{t-\tau_3}^{t} [\int_{s}^{t} \hat{\psi}_3(\hat{e}(\theta))^T\ d\theta] Z_2 [\int_{s}^{t} \hat{\psi}_3(\hat{e}(\theta))\ d\theta]\ ds + \int_{0}^{\tau_3}\int_{t-s}^{t} (\theta - t + s)\hat{\psi}_3(\hat{e}(\theta))^T Z_2 \hat{\psi}_3(\hat{e}(\theta))\ d\theta\, ds,$$
$$V_6(t) = \int_{t-\tau_3}^{t} [\int_{s}^{t} \hat{e}(\theta)^T\ d\theta] U_2 [\int_{s}^{t} \hat{e}(\theta)\ d\theta]\ ds + \int_{0}^{\tau_3}\int_{t-s}^{t} (\theta - t + s)\hat{e}(\theta)^T U_2 \hat{e}(\theta)\ d\theta\, ds. \tag{7a-f}$$

Derivatives of $V_i(t)$, $i = 1, \cdots, 6$, are given, respectively, by

$$\dot{V}_1(t) = 2\dot{\hat{e}}(t)^T [P_1 \hat{e}(t) + \int_{-\tau_1}^{0} Q(\xi)\hat{e}(t+\xi)\ d\xi] + 2\hat{e}(t)^T \int_{-\tau_1}^{0} Q(\xi)\dot{\hat{e}}(t+\xi)\ d\xi$$
$$\dot{V}_2(t) = \hat{\psi}_2(\hat{e}(t))^T Z_1 \hat{\psi}_2(\hat{e}(t)) + \dot{\hat{e}}(t)^T U_1 \dot{\hat{e}}(t)$$
$$- \hat{\psi}_2(\hat{e}(t-\tau_1))^T Z_1 \hat{\psi}_2(\hat{e}(t-\tau_1)) - \dot{\hat{e}}(t-\tau_2)^T U_1 \dot{\hat{e}}(t-\tau_2)$$
$$\dot{V}_3(t) = 2\int_{-\tau_1}^{0} \dot{\hat{e}}(t+\xi)^T S(\xi)\hat{e}(t+\xi)\ d\xi + 2\int_{-\tau_1}^{0}\int_{-\tau_1}^{0} \dot{\hat{e}}(t+s)^T R(s,\xi)\hat{e}(t+\xi)\ ds\, d\xi$$
$$\dot{V}_4(t) = 2\int_{-\tau_2}^{0} \dot{\hat{e}}(t+\xi)^T H(\xi)\hat{e}(t+\xi)\ d\xi + 2\int_{-\tau_2}^{0}\int_{-\tau_2}^{0} \dot{\hat{e}}(t+s)^T T(s,\xi)\hat{e}(t+\xi)\ ds\, d\xi$$
$$\dot{V}_5(t) \le \tau_3^2\, \hat{\psi}_3(\hat{e}(t))^T Z_2 \hat{\psi}_3(\hat{e}(t)) - e^{-2\alpha\tau_3} [\int_{t-\tau_3}^{t} e^{\alpha(t-\theta)} \hat{\psi}_3(\hat{e}(\theta))^T\ d\theta] Z_2 [\int_{t-\tau_3}^{t} e^{\alpha(t-\theta)} \hat{\psi}_3(\hat{e}(\theta))\ d\theta]$$
$$\dot{V}_6(t) \le \tau_3^2\, \hat{e}(t)^T U_2 \hat{e}(t) - [\int_{t-\tau_3}^{t} \hat{e}(\theta)^T\ d\theta] U_2 [\int_{t-\tau_3}^{t} \hat{e}(\theta)\ d\theta] \tag{8a-f}$$

According to Remark 1, we have

$$-(\hat{\psi}_{1i}(\hat{e}_i(t)) - f_i^+ \hat{e}_i(t))^T (\hat{\psi}_{1i}(\hat{e}_i(t)) - f_i^- \hat{e}_i(t)) \ge 0, \tag{9a}$$
$$-(\hat{\psi}_{2i}(\hat{e}_i(t-\tau_1)) - g_i^+ \hat{e}_i(t-\tau_1))^T (\hat{\psi}_{2i}(\hat{e}_i(t-\tau_1)) - g_i^- \hat{e}_i(t-\tau_1)) \ge 0, \tag{9b}$$
$$-(\hat{\psi}_{3i}(\hat{e}_i(t)) - h_i^+ \hat{e}_i(t))^T (\hat{\psi}_{3i}(\hat{e}_i(t)) - h_i^- \hat{e}_i(t)) \ge 0, \tag{9c}$$

which are, respectively, equivalent to

$$\overleftrightarrow{\psi}_1(t)^T \Delta_{f_i} \overleftrightarrow{\psi}_1(t) \ge 0, \tag{10a}$$
$$\overleftrightarrow{\psi}_2(t-\tau_1)^T \Delta_{g_i} \overleftrightarrow{\psi}_2(t-\tau_1) \ge 0, \tag{10b}$$
$$\overleftrightarrow{\psi}_3(t)^T \Delta_{h_i} \overleftrightarrow{\psi}_3(t) \ge 0, \tag{10c}$$

where $\overleftrightarrow{\psi}_i(t) := [\hat{e}(t)^T, \hat{\psi}_i(\hat{e}(t))^T]^T$ and

$$\Delta_{f_i} := \begin{bmatrix} -f_i^+ f_i^- v_i v_i^T & \dfrac{f_i^+ + f_i^-}{2} v_i v_i^T \\ * & -v_i v_i^T \end{bmatrix}, \Delta_{g_i} := \begin{bmatrix} -g_i^+ g_i^- v_i v_i^T & \dfrac{g_i^+ + g_i^-}{2} v_i v_i^T \\ * & -v_i v_i^T \end{bmatrix},$$

$$\Delta_{h_i} := \begin{bmatrix} -h_i^+ h_i^- v_i v_i^T & \dfrac{h_i^+ + h_i^-}{2} v_i v_i^T \\ * & -v_i v_i^T \end{bmatrix}.$$

Moreover, from (3) and (4), the following equation holds for any matrices P_2, P_3 with appropriate dimensions:

$$\begin{aligned} &2(\hat{e}(t)^T P_2^T + \dot{\hat{e}}(t)^T P_3^T)(-\dot{\hat{e}}(t) - (A + K_1 - \alpha I)\hat{e}(t) - K_2 \hat{e}(t-\tau_1) \\ &+ W_1 \hat{\psi}_1(\hat{e}(t)) + W_2 e^{\alpha\tau_1} \hat{\psi}_2(\hat{e}(t-\tau_1)) - \alpha W_3 e^{\alpha\tau_2} \hat{e}(t-\tau_2) \\ &+ W_3 e^{\alpha\tau_2} \dot{\hat{e}}(t-\tau_2) - K_3 \int_{t-\tau_3}^{t} \hat{e}(s)\, ds + W_4 \int_{t-\tau_3}^{t} e^{\alpha(t-s)} \hat{\psi}_3(\hat{e}(s))\, ds - E\,\hat{w}(t)) = 0 \end{aligned} \quad (11)$$

According to [43], the delay intervals $[-\tau_1, 0]$ and $[-\tau_2, 0]$ are, respectively, divided into N segments $[\theta_p, \theta_{p-1}]$ and $[\hat{\theta}_p, \hat{\theta}_{p-1}]$, $p = 1, \cdots, N$ of equal length, i.e. $h_i = \tau_i / N$, where $\theta_p = -p\,h_1$ and $\hat{\theta}_p = -p\,h_2$. In the sequel, the continuous functions $Q(.), S(.), H(.)$, $R(.,.)$ and $T(.,.)$ are chosen to be piecewise linear, i.e., $Q(\theta_p + \kappa h_1) = (1-\kappa) Q_p + \kappa Q_{p-1}$, $S(\theta_p + \kappa h_1) = (1-\kappa) S_p + \kappa S_{p-1}$, $H(\hat{\theta}_p + \kappa h_2) = (1-\kappa) H_p + \kappa H_{p-1}$,

$$R(\theta_p + \kappa h_1, \theta_q + \beta h_1) = \begin{cases} (1-\kappa) R_{pq} + \beta R_{p-1,q-1} + (\kappa - \beta) R_{p-1,q}, & \kappa \ge \beta \\ (1-\beta) R_{pq} + \kappa R_{p-1,q-1} + (\beta - \kappa) R_{p,q-1}, & \kappa < \beta \end{cases}$$

$$T(\hat{\theta}_p + \kappa h_2, \hat{\theta}_q + \beta h_2) = \begin{cases} (1-\kappa) T_{pq} + \beta T_{p-1,q-1} + (\kappa - \beta) T_{p-1,q}, & \kappa \ge \beta \\ (1-\beta) T_{pq} + \kappa T_{p-1,q-1} + (\beta - \kappa) T_{p,q-1}, & \kappa < \beta \end{cases}$$

with $\dot{S}(\xi) = h_1^{-1}(S_{p-1} - S_p)$, $\dot{Q}(\xi) = h_1^{-1}(Q_{p-1} - Q_p)$, $\dot{H}(\xi) = h_2^{-1}(H_{p-1} - H_p)$, $\frac{\partial}{\partial \xi} R(\xi, \theta) + \frac{\partial}{\partial \theta} R(\xi, \theta) = h_1^{-1}(R_{p-1,q-1} - R_{p,q})$ and $\frac{\partial}{\partial \xi} T(\xi, \theta) + \frac{\partial}{\partial \theta} T(\xi, \theta) = h_2^{-1}(T_{p-1,q-1} - T_{p,q})$. Thus, one obtains

$$\begin{aligned} &2\dot{\hat{e}}(t)^T \int_{-\tau_1}^{0} Q(\xi)\, \hat{e}(t+\xi)\, d\xi \\ &= 2\dot{\hat{e}}(t)^T \sum_{p=1}^{N} h_1 \int_0^1 [Q_p^s + (1-2\kappa) Q_p^a]\, \hat{e}(t + \theta_p + \kappa h_1)\, d\kappa \end{aligned} \quad (12a)$$

$$\begin{aligned} &2\hat{e}(t-\tau_1)^T \int_{-\tau_1}^{0} R(-\tau_1, \xi)\, \hat{e}(t+\xi)\, d\xi \\ &= 2\hat{e}(t-\tau_1)^T \sum_{p=1}^{N} h_1 \int_0^1 [R_{N,p}^s + (1-2\kappa) R_{N,p-1}^a]\, \hat{e}(t + \theta_p + \kappa h_1)\, d\kappa \end{aligned} \quad (12b)$$

$$\begin{aligned} &2\hat{e}(t-\tau_2)^T \int_{-\tau_2}^{0} T(-\tau_2, \xi)\, \hat{e}(t+\xi)\, d\xi = \\ &2\hat{e}(t-\tau_2)^T \sum_{p=1}^{N} h_2 \int_0^1 [T_{N,p}^s + (1-2\kappa) T_{N,p-1}^a]\, \hat{e}(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \end{aligned} \quad (12c)$$

$$\int_{-\tau_1}^{0} \hat{e}(t+\xi)^T \dot{S}(\xi)\, \hat{e}(t+\xi)\, ds = \sum_{p=1}^{N} \int_0^1 \hat{e}(t + \theta_p + \kappa h_1)^T (S_{p-1} - S_p)\, \hat{e}(t + \theta_p + \kappa h_1)\, d\kappa \quad (12d)$$

$$\int_{-\tau_2}^{0} \hat{e}(t+\xi)^T \dot{H}(\xi)\, \hat{e}(t+\xi)\, ds = \sum_{p=1}^{N} \int_0^1 \hat{e}(t + \hat{\theta}_p + \kappa h_2)^T (H_{p-1} - H_p)\, \hat{e}(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \quad (12e)$$

$$\begin{aligned} &\int_{-\tau_1}^{0} \int_{-\tau_1}^{0} \hat{e}(t+s)^T \left(\frac{\partial}{\partial s} R(s,\xi) + \frac{\partial}{\partial \xi} R(s,\xi)\right) \hat{e}(t+\xi)\, ds\, d\xi \\ &= h_1 \sum_{q=1}^{N} \sum_{p=1}^{N} \int_0^1 \hat{e}(t + \theta_p + \beta h_1)^T (R_{p-1,q-1} - R_{p,q})\, \hat{e}(t + \theta_p + \kappa h_1)\, d\kappa\, d\beta \end{aligned} \quad (12f)$$

$$\begin{aligned} &\int_{-\tau_2}^{0} \int_{-\tau_2}^{0} \hat{e}(t+s)^T \left(\frac{\partial}{\partial s} T(s,\xi) + \frac{\partial}{\partial \xi} T(s,\xi)\right) \hat{e}(t+\xi)\, ds\, d\xi = \\ &h_2 \sum_{q=1}^{N} \sum_{p=1}^{N} \int_0^1 \hat{e}(t + \hat{\theta}_p + \beta h_2)^T (T_{p-1,q-1} - T_{p,q})\, \hat{e}(t + \hat{\theta}_p + \kappa h_2)\, d\kappa\, d\beta \end{aligned} \quad (12g)$$

$$\begin{aligned} &2\hat{e}(t)^T \int_{-\tau_1}^{0} (-\dot{Q}(\xi) + R(0,\xi))\, \hat{e}(t+\xi)\, d\xi \\ &= 2h_1 \hat{e}(t)^T \sum_{p=1}^{N} \int_0^1 (2Q_p^a + R_{0,p}^s + (1-2\kappa) R_{0,p}^a)\, \hat{e}(t + \theta_p + \kappa h_1)\, d\kappa \end{aligned} \quad (12h)$$

$$\begin{aligned} &2\hat{e}(t)^T \int_{-\tau_2}^{0} T(0,\xi)\, \hat{e}(t+\xi)\, d\xi = \\ &2h_2 \hat{e}(t)^T \sum_{p=1}^{N} \int_0^1 (T_{0,p}^s + (1-2\kappa) T_{0,p}^a)\, \hat{e}(t + \hat{\theta}_p + \kappa h_2)\, d\kappa \end{aligned} \quad (12i)$$

Now, from (3), using the obtained derivative terms in (8) and (10)-(12), we obtain the following result,

$$\begin{aligned} &\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t) \le \chi_e^T(t)(\Xi_e + D^s \widetilde{U}_1 D^{s^T} \\ &+ O^s \widetilde{U}_2 O^{s^T} + \frac{1}{3}(D^a \widetilde{U}_1 D^{a^T} + O^a \widetilde{U}_2 O^{a^T}))\chi_e(t) \\ &- \int_0^1 \phi_e(\kappa;\alpha)^T d\kappa R_{ds} \int_0^1 \phi_e(\kappa;\alpha)\, d\kappa - \int_0^1 \hat{\phi}_e(\kappa;\alpha)^T d\kappa T_{ds} \int_0^1 \hat{\phi}_e(\kappa;\alpha)\, d\kappa \\ &- \int_0^1 \phi_D(\kappa;\alpha)^T \Theta_1\, \phi_D(\kappa;\alpha)\, d\kappa - \int_0^1 \phi_O(\kappa;\alpha)^T \Theta_2\, \phi_O(\kappa;\alpha)\, d\kappa \end{aligned} \quad (13)$$

with $\chi_e(t) = col\{\hat{e}(t), \dot{\hat{e}}(t), \hat{e}(t-\tau_1), \hat{e}(t-\tau_2), \dot{\hat{e}}(t-\tau_2), \hat{\psi}_1(\hat{e}(t)), \hat{\psi}_2(\hat{e}(t)), \hat{\psi}_3(\hat{e}(t)), \int_{t-\tau_3}^{t} \hat{\psi}_3(\hat{e}(\theta))\, d\theta, \hat{\psi}_2(\hat{e}(t-\tau_1)), \hat{w}(t))\}$, $\chi_D(t) := (D^s + (1-2\kappa) D^a)^T \times \chi_e(t)$, $\chi_O(t) := (O^s + (1-2\kappa) O^a)^T \chi_e(t)$, $\phi_D(\kappa;\alpha) := [\chi_D(t)^T, \phi_e(\kappa;\alpha)^T]^T$, $\phi_O(\kappa;\alpha) := [\chi_O(t)^T, \hat{\phi}_e(\kappa;\alpha)^T]^T$ and

$$P = \begin{bmatrix} P_1 & 0 \\ P_2 & P_3 \end{bmatrix}, \ \Theta_1 := \begin{bmatrix} \widetilde{U}_1 & -I \\ * & S_d \end{bmatrix}, \Theta_2 := \begin{bmatrix} \widetilde{U}_2 & -I \\ * & H_d \end{bmatrix},$$

$$\Xi_e = \begin{bmatrix} \widetilde{\Sigma}_{11} & \begin{bmatrix} P_2^T K_2 - Q_N + C_1^T C_2 \\ P_3^T K_2 \end{bmatrix} & \begin{bmatrix} 0 \\ -\alpha P_3^T W_3 e^{\alpha\tau_2} \end{bmatrix} & \begin{bmatrix} P_2^T W_3 \\ P_3^T W_3 \end{bmatrix} e^{\alpha\tau_2} & \widetilde{\Sigma}_{15} \\ * & -S_N + C_2^T C_2 & 0 & 0 & 0 \\ * & * & -H_N & 0 & 0 \\ * & * & * & -U & 0 \\ * & * & * & * & \Sigma_{55} \end{bmatrix},$$

$$\widetilde{\Sigma}_{11} = sym(P^T \begin{bmatrix} 0 & I \\ -(A - K_1 - \alpha I) & -I \end{bmatrix}) + \begin{bmatrix} \begin{pmatrix} sym(Q_0) + S_0 + H_0 - F^+ \Lambda_1 F^- \\ -G^+ \Lambda_2 G^- - H^+ \Lambda_3 H^- + \tau_3^2 U_2 \end{pmatrix} & 0 \\ * & U_1 \end{bmatrix}$$

$$\widetilde{\Sigma}_{15} = \left[\begin{matrix} \begin{bmatrix} P_2^T W_1 + \frac{1}{2}(F^+ + F^-)\Lambda_1 \\ D^T + P_3^T W_1 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(G^+ + G^-)\Lambda_2 \\ 0 \end{bmatrix} & \begin{bmatrix} \frac{1}{2}(H^+ + H^-)\Lambda_3 \\ 0 \end{bmatrix} \\ 0 & 0 & 0 \end{matrix}\right.$$

$$\left.\begin{matrix} \begin{bmatrix} P_2^T W_4 + C_1^T C_3 \\ P_3^T W_4 \end{bmatrix} & \begin{bmatrix} P_2^T W_2 \\ P_3^T W_2 \end{bmatrix} & \begin{bmatrix} -P_2^T E \\ -P_3^T E \end{bmatrix} \\ 0 & 0 & 0 \end{matrix}\right]$$

with $\phi_e(\kappa;\alpha) = col\{\hat{e}(t + \theta_1 + \kappa h_1), \hat{e}(t + \theta_2 + \kappa h_1), \cdots, \hat{e}(t + \theta_N + \kappa h_1)\}$,

$\hat{\phi}_e(\kappa;\alpha) = col\{\hat{e}(t + \hat{\theta}_1 + \kappa h_1),\ \hat{e}(t + \hat{\theta}_2 + \kappa h_1), \cdots, \hat{e}(t + \hat{\theta}_N + \kappa h_1)\}$.

By assuming $\overline{U}_i = \widetilde{U}_i^{-1}$, from (5b)-(5c) it can be easily seen that $\Theta_i > 0,\ (i=1,2)$. Then applying the Jensen's inequality [26] to the forth and fifth terms in the right side of (13), one finds

$$\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t) \le \widetilde{\chi}_e(t)^T\, \widetilde{\Xi}_e\, \widetilde{\chi}_e(t) \qquad (14)$$

where $\widetilde{\chi}_e(t) = [\chi_e(t)^T, \int_0^1 \phi_e(\kappa;\alpha)^T\, d\kappa, \int_0^1 \hat{\phi}_e(\kappa;\alpha)^T\, d\kappa]^T$ and

$$\widetilde{\Xi}_e = \begin{bmatrix} \Xi_e + \frac{1}{3}(D^a \widetilde{U}_1 D^{a^T} + O^a \widetilde{U}_2 O^{a^T}) & -D^s & -O^s \\ * & -S_d - R_{ds} & 0 \\ * & * & -H_d - T_{ds} \end{bmatrix}.$$

For a $\gamma > 0$ and under zero initial conditions, one has

$$J_\infty \le \int_0^\infty [\hat{z}_e(t)^T \hat{z}_e(t) - \gamma^2 \hat{w}(t)^T \hat{w}(t) + \dot{V}(t)]\ dt \qquad (15)$$

and $J_\infty < 0$ means that $\widetilde{\Xi}_e < 0$ satisfies the H_∞ performance measure, and by applying Schur complement, one gets

$$\begin{bmatrix} \Xi_e & D^s & O^s & D^a & O^a \\ * & -S_d - R_{ds} & 0 & 0 & 0 \\ * & * & -H_d - T_{ds} & 0 & 0 \\ * & * & * & -3\overline{U}_1 & 0 \\ * & * & * & * & -3\overline{U}_2 \end{bmatrix} < 0\,. \qquad (16)$$

Following [27] we choose $P_3 = \delta P_2$, $\delta \in R$, where δ is a tuning scalar parameter. Therefore, considering $L_i = P_2^T K_i$ results in the LMI (5d).

On the other hand, the condition $J_\infty < 0$ for $\hat{w}(t) \equiv 0$ implies $\dot{V}(t) < 0$. Then, we have $V(t) < V(0)$. From (6)-(7), one gets

$$\begin{aligned} V(0) \le\ & \lambda_{\max}(P_1)\|\zeta\|^2 + \tau_1 \lambda_{\max}(Q_p)_{p=0}^N \|\zeta\|^2 + \tau_1 \lambda_{\max}(G^{+^T} Z_1 G^+)\|\zeta\|^2 \\ & + \tau_2 \lambda_{\max}(U_1)\|\dot{\zeta}\|^2 + \tau_1 \lambda_{\max}(S_p)_{p=0}^N \|\zeta\|^2 + \tau_1^2 \lambda_{\max}(R_{p,p})_{p=0}^N \|\zeta\|^2 \\ & + \tau_2 \lambda_{\max}(H_p)_{p=0}^N \|\zeta\|^2 + \tau_2^2 \lambda_{\max}(T_{p,p})_{p=0}^N \|\zeta\|^2 \\ & + \frac{1}{2}\tau_3^3 (\lambda_{\max}(H^{+^T} Z_2 H^+) + \lambda_{\max}(U_2))\|\zeta\|^2 \\ =\ & \Delta_1 \|\zeta\|^2 + \Delta_2 \|\dot{\zeta}\|^2 \end{aligned} \qquad (17)$$

where $\Delta_1 = \lambda_{\max}(P_1) + \tau_1 \lambda_{\max}(Q_p)_{p=0}^N + \tau_1 \lambda_{\max}(G^{+^T} Z_1 G^+) + \tau_1 \lambda_{\max}(S_p)_{p=0}^N + \tau_1^2 \lambda_{\max}(R_{p,p})_{p=0}^N + \tau_2 \lambda_{\max}(H_p)_{p=0}^N + + \tau_2^2 \lambda_{\max}(T_{p,p})_{p=0}^N + \frac{1}{2}\tau_3^3(\lambda_{\max}(H^{+^T} Z_2 H^+) + \lambda_{\max}(U_2))$ and $\Delta_2 = \tau_2 \lambda_{\max}(U_1)$.

Moreover, $V(t) \ge e^{2\alpha t} e(t)^T P_1 e(t) \ge e^{2\alpha t} \lambda_{\min}(P_1)\|e(t)\|^T$. Therefore,

$$|e(t)| \le \sqrt{\frac{\max\{\Delta_1, \Delta_2\}}{\lambda_{\min}(\Sigma)}}\, e^{-\alpha t}[\|\zeta\| + \|\dot{\zeta}\|]$$

which shows that the synchronization error network (3) with (4) is globally exponentially stable and has the exponential convergence rate *k*. This completes the proof. ■

IV. Numerical Example

Consider the MSNNs (1)-(2) with the following matrices adopted from [6, 10]:

$$A = I_2,\ W_1 = \begin{bmatrix} 1+\pi/4 & 20 \\ 0.1 & 1+\pi/4 \end{bmatrix},\ W_2 = \begin{bmatrix} -1.3\sqrt{2}\,\pi/4 & 0.1 \\ 0.1 & -1.3\sqrt{2}\,\pi/4 \end{bmatrix},$$

$$W_3 = 0.1 I_2,\ W_4 = \begin{bmatrix} 2+\pi/2 & 40 \\ 0.2 & 2+\pi/2 \end{bmatrix},\ C_1 = C_2 = C_3 = 1,\ E = [1 \quad 1]^T,$$

$$\tau_1 = 1, \tau_2 = 0.5, \tau_3 = 0.2, \phi(t) = [0.3 \quad -0.7], \varphi(t) = [0 \quad 0],$$

$$f(x_i(t)) = g(x_i(t)) = h(x_i(t)) = 0.5(|x_i(t)+1| - |x_i(t)-1|)$$

Table 1

$\gamma_{optimal}$ Comparision w.r.t. N and α.

	$\alpha = 0.5$	$\alpha = 1$	$\alpha = 1.5$
$N = 1$	0.3065	0.3250	0.3345
$N = 2$	0.2920	0.3205	0.3285
$N = 3$	0.2895	0.3165	0.3215

In light of Theorem 1, LMIs (5) using Matlab LMI Control Toolbox for different values of parameter N, i.e. $N \in \{1,2,3\}$, and different values of the exponential synchronization degree α, i.e. $\alpha \in \{0.5, 1, 1.5\}$, are solved and the values of the parameter $\gamma_{optimal}$ are obtained and shown in Table 1. It is easily seen that for a fixed value of the exponential synchronization degree α, $\gamma_{optimal}$ is decreased as the parameter N is increased and for a fixed value of the parameter N, $\gamma_{optimal}$ is increased as the parameter α is increased. Moreover, the control gains K_i for $\alpha = 0.5$ are calculated and illustrated in Table 2.

Table 2

Controller gains (with $\alpha = 0.5$) w.r.t. N.

	$N = 1$	$N = 2$	$N = 3$
K_1	$10^3 \cdot \begin{bmatrix} 1.2784 & 0.3765 \\ 0.5684 & 1.2305 \end{bmatrix}$	$10^3 \cdot \begin{bmatrix} 1.7759 & 0.3182 \\ 0.3237 & 0.2737 \end{bmatrix}$	$10^3 \cdot \begin{bmatrix} 2.5903 & 1.0947 \\ 1.3883 & 2.3299 \end{bmatrix}$
K_2	$\begin{bmatrix} -0.2117 & 0.2374 \\ 0.1541 & -0.2781 \end{bmatrix}$	$\begin{bmatrix} 0.5864 & 0.0560 \\ 0.0676 & 0.2468 \end{bmatrix}$	$\begin{bmatrix} -1.0150 & -0.4911 \\ -0.3406 & -0.8680 \end{bmatrix}$
K_3	$\begin{bmatrix} -0.0965 & -0.1318 \\ -0.1093 & -0.1201 \end{bmatrix}$	$\begin{bmatrix} 0.2450 & 0.3790 \\ 0.4570 & 0.8645 \end{bmatrix}$	$\begin{bmatrix} 0.0864 & 0.0560 \\ 0.0676 & 0.0468 \end{bmatrix}$

V. Conclusion

This paper presented the exponential H_∞ synchronization problem for master and slave neural networks (MSNNs) with mixed time-delays, where the mixed delays comprise different neutral, discrete and distributed time-delays and the class covers the Lipschitz-type nonlinearity case. An appropriate discretized Lyapunov-Krasovskii functional and some free weighting matrices were utilized to establish some delay-dependent sufficient conditions for designing a delayed state-feedback control as a synchronization law by convex optimization over linear matrix inequalities (LMIs). It was shown that the synchronization law guaranteed the exponential H_∞ synchronization of the two coupled MSNNs regardless of their initial states. Detailed comparisons with different number of segments were made and numerical simulations were carried out to demonstrate the effectiveness of the established synchronization laws.

ACKNOWLEDGMENT

This research was partially funded by the German Research Foundation (DFG) as part of the Collaborative Research Center 637 '*Autonomous Cooperating Logistic Processes: A Paradigm Shift and its Limitations*' (SFB 637).

REFERENCES

[1] Chen G., '*Control and Synchronization of Chaos, a Bibliography*' Dept. of Electrical Engineering, University of Houston, Texas, 1997.
[2] Chen G. and Dong X., 'On feedback control of chaotic continuous-time systems,' *IEEE Trans. Circuits and Systems*, 40, 591–601, 1993.
[3] Pecora L.M. and Carroll T.L., 'Synchronization in chaotic systems,' *Physical Review Letters,* 64, 821–824, 1990.
[4] Kapitaniak T., 'Continuous control and synchronization in chaotic systems,' *Chaos Solitons and Fractals*, 6, 237–244, 1995.
[5] Gao H., Lam J., and Chen G., 'New criteria for synchronization stability of general complex dynamical networks with coupling delays' Physics Letters A , 360(2), 263-273, 2006.
[6] Cheng C. J., Liao T. L., Yan J. J. and Hwang C. C., 'Exponential synchronization of a class of neural networks with time-varying delays' *IEEE Trans. Sys., Man, and Cyber., Part B: Cybernetics*, 36(1), 209-215, 2006.
[7] Lu H., and van Leeuwen C., 'Synchronization of chaotic neural networks via output or state coupling' *Chaos, Solitons and Fractals*, 30, 166-176, 2006.
[8] Hou Y. Y., Liao T. L., Yan J. J.' H_∞ synchronization of chaotic systems using output feedback control design' *Physics A*, 379, 81-89, 2007.
[9] Wen G., Wang Q.G., Lin C., Han X., Li G., 'Synthesis for robust synchronization of chaotic systems under output feedback control with multiple random delays' *Chaos, Solitons & Fractals*, 29(5), 1142-1146, 2006.
[10] Karimi H.R., Maass P., 'Delay-range-dependent exponential H_∞ synchronization of a class of delayed neural networks' *Chaos, Solit. & Fractals*, doi:10.1016/j.chaos.2008.04.051.
[11] Femat R., Alvarez-Ramírez J., Fernández-Anaya G., 'Adaptive synchronization of high-order chaotic systems: a feedback with low-order parametrization' *Physica D: Nonlinear Phenomena*, 139, 231-246, 2000.
[12] Liao T. L., and Tsai S. H., 'Adaptive synchronization of chaotic systems and its application to secure communication' *Chaos, Solitons and Fractals*, 11(9), 1387-1396, 2000.
[13] Feki M., 'An adaptive chaos synchronization scheme applied to secure communication' *Chaos, Solit. & Fractals*, 18, 141-148, 2003.
[14] Salarieh H. and Shahrokhi M., 'Indirect adaptive control of discreet chaotic systems,' *Chaos Solitons and Fractals*, 34, 1188–1201, 2007.
[15] Wang Y.W., Changyun W., Yeng C.S. and Xiao J.W., 'Adaptive control and synchronization for a class of nonlinear chaotic systems using partial system states,' *Physics letters. A*, 351(1-2), 79-84, 2006.
[16] Park J.H., 'Synchronization of Genesio chaotic system via backstepping approach' *Chaos Solit. & Fractals,* 27, 1369–1375, 2006.
[17] Yan J.J., Yang Y.S., Chiang T.Y. and Chen C.Y., 'Robust synchronization of unified chaotic systems via sliding mode control' *Chaos, Solitons & Fractals*, 34(3), 947-954, 2007.
[18] Behzad M., Salarieh H., and Alasty A., 'Chaos synchronization in noisy environment using nonlinear filtering and sliding mode control' *Chaos Solitons and Fractals,* 36, 1295-1304, 2008.
[19] Yan J.J., Hung M.L., Chiang T.Y. and Yang Y.S., 'Robust synchronization of chaotic systems via adaptive sliding mode control' *Physics Letters A*, 356(3), 220-225, 2006.
[20] García-Valdovinos L.-G.**,** Parra-Vega V. and Arteaga M.A., **'**Observer-based sliding mode impedance control of bilateral teleoperation under constant unknown time delay' *Robotics and Automation Systems*, 55(8), 609-617, 2007.
[21] Li W.L., Chang K.M., 'Robust synchronization of drive–response chaotic systems via adaptive sliding mode control' *Chaos, Solitons & Fractals*, Available online 6 August 2007 (In Press).
[22] Cai C., and Chen G., 'Synchronization of complex dynamical networks by the incremental ISS approach' *Physica A*, 371, 754-766, 2006.
[23] Niculescu S. I., 2001, *Delay Effects on Stability: A Robust Control Approach* (Berlin: Springer).
[24] Fridman, E., 'New Lyapunov-Krasovskii functionals for stability of linear retarded and neutral type systems.' *Systems* & *Control Letters*, 43, 309-319, 2001.
[25] Lam, J., Gao, H., Wang, C., ' H_∞ model reduction of linear systems with distributed delay.' *IEE Proc. Contr. Theory Appl*., 152(6), 662-674, 2005.
[26] Park, P., 'A delay-dependent stability criterion for systems with uncertain time-invariant delays.' *IEEE Trans. Automatic Control*, 44, 876-877, 1999.
[27] Fridman, E., and Shaked, U., 'A descriptor system approach to H_∞ control of linear time-delay systems.' *IEEE Trans. Automatic control*, 47(2), 253-270, 2002.
[28] Wang Z., Lauria S., Fang J. and Liu X., 'Exponential stability of uncertain stochastic neural networks with mixed time-delays' *Chaos, Solitons and Fractals*, 32, 62-72, 2007.
[29] He Y., Wang Q., Wu M., Lin C., 'Delay-dependent state estimation for delayed neural networks' *IEEE Trans. Neur. Net.*, 17, 1077-1081, 2006.
[30] Lu H., 'Comments on 'A generalized LMI-based approach to the global asymptotic stability of delayed cellular neural networks'', *IEEE Trans. Neural Networks*, 16, 778-779, 2005.
[31] Zhang Q., X. Wen, and J. Xu, 'Delay-dependent exponential stability of cellular neural networks with time-varying delays' *Chaos, Solitons Fractals*, 23, 1363-1369, 2005.
[32] Wang Z., Liu Y. and Liu X., 'On global asymptotic stability of neural networks with discrete and distributed delays', *Physics Letters A*, 345, 299-308, 2005.
[33] Xu S., Lam J., and Ho D. W. C., and Zou Y., 'Novel global asymptotic stability criteria for delayed cellular neural networks' *IEEE Trans. Circuits and Systems—II: Express Briefs*, 52(6), 349-353, 2005.
[34] Xu S., Lam J., and Ho D. W. C., "A new LMI condition for delay-dependent asymptotic stability of delayed Hopfield neural networks," *IEEE Trans. Circuits Syst. II. Exp. Briefs*, 53(3), 230–234, Mar. 2006.
[35] Lou X. Y. and Cui B., "New LMI conditions for delay-dependent asymptotic stability of delayed Hopfield neural networks," *Neurocomputing*, 69(16–18), 2374–2378, 2006.
[36] He Y., Wang Q., and Zheng W., "Global robust stability for delayed neural networks with polytopic type uncertainties," *Chaos Solitons Fractals*, 26, 1349–1354, 2005.
[37] He Y., Wu M., and She J. H., "An improved global asymptotic stability criterion for delayed cellular neural networks," *IEEE Trans. Neural Netw.*, 17(1), 250–252, 2006.
[38] Mou S., Gao H., Lam J., and Qiang W., 'A new criterion of delay-dependent asymptotic stability for Hopfield neural networks with time delay' *IEEE Trans. Neural Networks*, 19(3), 532-535, 2008.
[39] Mou S., Gao H., Qiang W., and Chen K., 'New delay-dependndet exponential stability for neural networks with time delay' IEEE Trans. Sys., Man and Cybernetics-Part B: Cybernetics, 38(2), 571-576, 2008.
[40] Gau R., Lien C., and Hsieh J., "Global exponential stability for uncertain cellular neural networks with multiple time-varying delays via LMI approach," *Chaos, Solit. & Fractals*, 32(4), 1258–1267, 2007.
[41] He Y., Wu M., and She J. H., "Delay-dependent exponential stability of delayed neural networks with time-varying delay," *IEEE Trans. Circuits Syst. II, Exp. Briefs*, 53(7), 553–557, 2006.
[42] Park J., "Further note on global exponential stability of uncertain cellular neural networks with variable delays," *Appl. Math. Comput.*, 188(1), 850–854, 2007
[43] Fridman E., 'Descriptor discretized Lyapunov functional method: Analysis and design**'** *IEEE Trans. Automatic Control*, 51(5), 890-897, 2006.