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Mixed H_2/H_∞ output-feedback control of second-order neutral systems with time-varying state and input delays

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Abstract

A mixed H_2/H_∞ output-feedback control design methodology is presented in this paper for second-order neutral linear systems with time-varying state and input delays. Delay-dependent sufficient conditions for the design of a desired control are given in terms of linear matrix inequalities (LMIs). A controller, which guarantees asymptotic stability and a mixed H_2/H_∞ performance for the closed-loop system of the second-order neutral linear system, is then developed directly instead of coupling the model to a first-order neutral system. A Lyapunov–Krasovskii method underlies the LMI-based mixed H_2/H_∞ output-feedback control design using some free weighting matrices. The simulation results illustrate the effectiveness of the proposed methodology.

Keywords: Second-order neutral systems; Mixed H_2/H_∞ output-feedback control; LMI; Time delay

1. Introduction

Delay differential systems represent a class of infinite-dimensional systems and are assuming an increasingly important role in many disciplines like economic, mathematics, science, and engineering (see for instance [1–4], and the references therein). For instance, in many control systems, delays appear either in the state, in the control input, or in the measurements. Therefore, how to analyze and synthesize dynamic systems with delayed arguments is a problem of recurring interest, as the delay may induce complex behavior (oscillation, instability, bad performances) for the systems concerned (see [2,5–7]). Neutral delay systems constitute a more general class than those of the retarded type. Stability of these systems proves to be a more complex issue because the system involves the derivative of the delayed state. Especially, in the past few decades increased attention has been devoted to the problem of robust delay-independent stability or delay-dependent stability and stabilization via different approaches for linear neutral systems with delayed state and/or input and parameter uncertainties (see for instance [2,8–12]). Among the existing results on neutral delay systems, the linear matrix inequality (LMI) approach is an efficient method to solve many control problems such as stability analysis and stabilization [13–16], H_∞ control problems [17–24], and guaranteed-cost (observer-based) control [25–31]. On the other hand, in spite of the fact that H_∞ controllers are robust with respect to the disturbances since they make no assumption about the disturbances, they have to accommodate for all conceivable disturbances, and are thus conservative. The mixed H_2/H_∞ control designs are quite useful for robust performance design for systems under parameter perturbations and uncertain disturbances. Recent works that employ robust mixed H_2/H_∞ state- and output-feedback control for neutral systems with time-varying delays have been completed, respectively, in References [32,33].

Second-order systems capture the dynamic behavior of many natural phenomena, and have found applications in many fields, such as vibrational and structural analysis, spacecraft control, electrical networks, robotics control and, hence, have attracted much attention (see, for instance, [34–43]). In the literature, a Haar-wavelet-based method for finite-time H_2

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control problem of the second-order retarded linear systems with respect to a quadratic cost function for any length of time is proposed in [44]. It is also worth citing that some appreciable pieces of work have been performed to design a guaranteed-cost control for the second-order neutral systems with time delay in control (see [45]). However, the system performance and stability are not investigated for a second-order neutral system in these works. Up until now, to the best of the authors' knowledge, no results about the delay-dependent mixed H_2/H_∞ output-feedback control of second-order neutral linear systems with time-varying state and input delays are available in the literature, which remains to be important and challenging. This motivates the present study.

In this paper, we make an attempt to develop an efficient approach for delay-dependent mixed H_2/H_∞ output-feedback control problem of second-order neutral linear systems with time-varying state and input delays. The main merit of the proposed method lies in the fact that it provides a convex problem via introduction of additional decision variables such that the control gains can be found from the LMI formulations without reformulating the system equations into a standard form of a first-order neutral system. By using a Lyapunov–Krasovskii method and some free weighting matrices, new sufficient conditions are established in terms of delay-dependent LMIs for the existence of desired mixed H_2/H_∞ output-feedback control such that the resulting closed-loop system is asymptotically stable and satisfies a prescribed mixed H_2/H_∞ performance. A significant advantage of our result is that the desired control is designed directly instead of coupling the model to a first-order neutral system and then applying the corresponding control designs in References [18,19,22,46] in a higher-dimensional space. Therefore, our result can be implemented in a numerically stable and efficient way for large-scale second-order neutral systems. Furthermore, as pointed out in [37], retaining the model in matrix second-order form has many advantages such as preserving physical insight of the original problem, preserving system matrix sparsity and structure, preserving uncertainty structure and entailing easier implementation (feedback control can be used directly). Finally, two numerical examples are given to illustrate the usefulness of our results.

The rest of this paper is organized as follows. Section 2 states the problem formulation and the needed assumptions and definitions. Section 3 includes the main results of the paper, that is, sufficient conditions for stability and mixed H_2/H_∞ performance, and delay-dependent mixed H_2/H_∞ output-feedback control design methodology. Section 4 provides two illustrative examples, and Section 5 concludes the paper.

Notations. The superscript 'T' stands for matrix transposition; $\Re^n$ denotes the n-dimensional Euclidean space; $\Re^{n\times m}$ is the set of all real m by n matrices. $\|\,.\,\|$ refers to the Euclidean vector norm or the induced matrix 2-norm. $\mathrm{col}\{\cdots\}$ and $\mathrm{diag}\{\cdots\}$ represent, respectively, a column vector and a block diagonal matrix and the operator $\mathrm{sym}(A)$ represents $A + A^{\mathrm{T}}$. $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote, respectively, the smallest and largest eigenvalues of the square matrix A. The notation $P > 0$ means that P is real symmetric and positive definite; the symbol $*$ denotes the elements below the main diagonal of a symmetric block matrix. In addition, $L_2^q[0,\infty)$ is adopted for the space of all functions $f : \Re \to \Re^q$ which are Lebesgue integrable in the square over $[0,\infty)$, with the standard norm $\|\,.\,\|_2$. Matrices, if the dimensions are not explicitly stated, are assumed to have compatible dimensions for algebraic operations.

2. Problem description

Many physical systems are modeled as second-order differential equations with delay. In the case of structural dynamics these are generally of the form

$$\begin{cases} M\ddot{x}(t) + M_1\ddot{x}(t-d(t)) + A\dot{x}(t) + A_1\dot{x}(t-r(t)) \\ \quad + Bx(t) + B_1x(t-r(t)) \\ \quad = Fu(t) + F_1u(t-h(t)) + Ew(t), & \text{(a)} \\ x(t) = \phi(t), \quad t \in [-\max\{h_M, d_M, r_M\}, 0], & \text{(b)} \\ \dot{x}(t) = \dot{\phi}(t), \quad t \in [-\max\{h_M, d_M, r_M\}, 0], & \text{(c)} \\ z(t) = C_1x(t) + C_2x(t-r(t)) + D_1u(t) \\ \quad + D_2u(t-h(t)), & \text{(d)} \\ y(t) = C_3x(t), & \text{(e)} \end{cases} \quad (1)$$

where $x(t) \in \Re^n$ is the state vector; $u(t) \in \Re^r$ is the control input; $w(t) \in L_2^q[0,\infty)$ is the external excitation (disturbance), $z(t) \in \Re^s$ is the controlled output and $y(t) \in \Re^l$ is the measured output. The coefficient matrices M, M_1, A, A_1, B and B_1 are square and real matrices, and the matrices F, F_1, E, C_1, C_2, C_3, D_1 and D_2 are real matrices with appropriate dimensions. The time-varying vector valued initial functions $\phi(t)$ and $\dot{\phi}(t)$ are continuously differentiable functionals, and the time-varying delays $h(t), d(t)$ and $r(t)$ are functions satisfying, respectively,

$$\begin{cases} 0 < d(t) \le d_M, \quad \dot{d}(t) \le d_D < 1, & \text{(a)} \\ 0 < h(t) \le h_M, \quad \dot{h}(t) \le h_D, & \text{(b)} \\ 0 < r(t) \le r_M, \quad \dot{r}(t) \le r_D. & \text{(c)} \end{cases} \quad (2)$$

The dynamical system (1) arises naturally in a wide range of applications, including: control of large flexible space structures, earthquake engineering; control of mechanical multi-body systems, stabilization of damped gyroscopic systems, robotics control, vibration control in structural dynamics, linear stability of flows in fluid mechanics and electrical circuit simulation (see e.g. [38–42] and the many references therein). In mechanical systems pairs of the matrices (M, M_1), (A, A_1) and (B, B_1) correspond to the mass, damping, and stiffness matrices and $x(t)$ is the vector of generalized displacements. The matrices F and F_1 distribute the force input to the correct degrees of freedom (see [34–36]).

Remark 1. In the second-order neutral system, taking $x_1(t) = x(t)$, $x_2(t) = \dot{x}(t)$ and $\xi(t) = \mathrm{col}\{x_1(t), x_2(t)\}$ yields an augmented system model, i.e., a first-order neutral linear system:

$$M_e\dot{\xi}(t) = A_e\,\xi(t) + A_{1e}\,\xi(t-r(t)) + M_{1e}\dot{\xi}(t-d(t)) \\ \quad + F_eu(t) + F_{1e}u(t-h(t)) + E_ew(t)$$

where

$$M_e = \begin{bmatrix} I & 0 \\ 0 & M \end{bmatrix}, \qquad A_e = \begin{bmatrix} 0 & I \\ -B & -A \end{bmatrix},$$

$$A_{1e} = \begin{bmatrix} 0 & 0 \\ -B_1 & -A_1 \end{bmatrix}, \qquad M_{1e} = \begin{bmatrix} 0 & 0 \\ 0 & -M_1 \end{bmatrix},$$
$$F_e = \begin{bmatrix} 0 \\ F \end{bmatrix}, \qquad F_{1e} = \begin{bmatrix} 0 \\ F_1 \end{bmatrix}, \qquad E_e = \begin{bmatrix} 0 \\ E \end{bmatrix}.$$

It is easy to understand that the proposed methods in References [18,19,22,46] to find a suitable robust control for the above neutral delay system eventually involve manipulations of $2n$-dimensional matrices $M_e, A_e, A_{1e}, M_{1e}, F_e, F_{1e}, E_e$, and hence will increase the dimension and number of the LMI variables in comparison with our result in this paper.

Throughout the paper, the following assumption is needed to enable the application of Lyapunov's method for the stability of neutral systems [47]:

(A1) Let the difference operator $D : C([-\max\{h_M, d_M, r_M\}, 0], \Re^n) \to \Re^n$ given by $Dx_t = Mx(t) + M_1 x(t - d(t))$ be delay-independently stable with respect to all delays. A sufficient condition for (A1) is that:

(A2) All the eigenvalues of the matrix $M^{-1}M_1$ are inside the unit circle.

Definition 1. (i) The H_2 performance measure of system (1) is defined as

$$J_2 = \int_0^{\infty} [\xi^{\mathrm{T}}(t) S_1 \xi(t) + u^{\mathrm{T}}(t) S_2 u(t)]\mathrm{d}t,$$

where $w(t) \equiv 0$, $\xi(t) := \mathrm{col}\{x(t), \dot{x}(t)\}$ and constant matrices $S_1, S_2 > 0$ are given.

(ii) The H_∞ performance measure of system (1) is defined as

$$J_\infty = \int_0^{\infty} [z^{\mathrm{T}}(t) z(t) - \gamma^2 w^{\mathrm{T}}(t) w(t)]\mathrm{d}t,$$

where the positive scalar γ is given.

(iii) The mixed H_2/H_∞ performance measure of system (1) is defined as

$$\mathrm{Min}\{J_0 | J_\infty < 0 \text{ and } J_2 \le J_0\}$$

or the so-called problem of minimizing an upper bound of J_2, i.e., $J_0 > 0$, under the constraint $J_\infty < 0$.

The problem to be addressed in this paper is formulated as follows: given the second-order neutral linear system (1) with time-varying delays (2) and a prescribed level of disturbance attenuation $\gamma > 0$, find a mixed H_2/H_∞ output-feedback control $u(t)$ of the form

$$u(t) = K_1 y(t) + K_2 \dot{y}(t) := KC\xi(t) \tag{3}$$

where $K := [\,K_1 \quad K_2\,]$, $C := \mathrm{diag}\{C_3, C_3\}$ and the matrices K_1 and K_2 are the control gains to be determined such that:

(1) the resulting closed-loop system (1) with (3) is asymptotically stable for any time delays satisfying (2);
(2) under $w(t) \equiv 0$, the H_2 performance measure satisfies $J_2 \le J_0$, where the positive scalar J_0 is said to be a guaranteed cost;
(3) under zero initial conditions and for all non-zero $w(t)$, the upper bound of the H_2 performance measure, i.e.,J_0, satisfies $J_\infty < 0$ (or the induced L_2-norm of the operator form $w(t)$ to the controlled outputs $z(t)$ is less than γ);
(4) in this case, the second-order neutral linear system (1) with (3) is said to be robustly asymptotically stable with a mixed H_2/H_∞ performance measure.

Remark 2. In this paper, the mixed H_2/H_∞ output-feedback control problem consists of the minimization of an upper bound of the H_2-norm of the system while a prescribed H_∞ attenuation level is guaranteed, allowing us to make a trade-off between the performance of the H_2 control and that of the H_∞ control. Up until now, several approaches have been proposed to solve the mixed H_2/H_∞ control problem: a Nash game theoretic approach was proposed to solve the mixed H_2/H_∞ control problem of deterministic linear systems through a set of cross-coupled Riccati equations in [48]. The method used in [48] has been generalized to the nonlinear [49], output-feedback control [50] and the stochastic systems governed by Itô differential equations with state-dependent noise [51–53].

Remark 3. It is noted that the second-order neutral system (1) is controlled by a proportional and derivative (PD) mixed H_2/H_∞ output-feedback control which has a direct application in the control of artificial satellites using motor driven inertia wheels as a source of torque (see for instance [36]). When $\mathrm{rank}(M) < n$, both the open-loop system (1) and the closed-loop system (1) by (3) are singular ones. For this case, the control of system (1) via the feedback control law (3) is equivalent to the output-feedback control in the first-order descriptor neutral linear system [43].

3. Main results

In this section, sufficient conditions for the solvability of the robust mixed H_2/H_∞ output-feedback control design problem are proposed using the Lyapunov method and an LMI approach. Before proceeding further, we give two technical lemmas, which are useful in the proof our main results.

Lemma 1 *([54]). For any arbitrary column vectors $a(s), b(s) \in \Re^p$, and any matrix $W \in \Re^{p\times p}$ and positive-definite matrix $H \in \Re^{p\times p}$ the following inequality holds:*

$$-2\int_{t-r(t)}^{t} b(s)^{\mathrm{T}} a(s)\mathrm{d}s \le \int_{t-r(t)}^{t} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix}^{\mathrm{T}} \times \begin{bmatrix} H & HW \\ * & (HW+I)^{\mathrm{T}} H^{-1} (HW+I) \end{bmatrix} \begin{bmatrix} a(s) \\ b(s) \end{bmatrix} \mathrm{d}s. \tag{4}$$

Lemma 2 *([55]). For a given $\mathrm{M} \in \Re^{p\times n}$ with $\mathrm{rank}(\mathrm{M}) = p < n$, assume that $Z \in \Re^{n\times n}$ is a symmetric matrix, then there exists a matrix $\hat{Z} \in \Re^{p\times p}$ such that $\mathrm{M}Z = \hat{Z}\mathrm{M}$ if and only if*

$$Z = V \begin{bmatrix} Z_1 & 0 \\ 0 & Z_2 \end{bmatrix} V^{\mathrm{T}},$$
$$\hat{Z} = U\hat{\mathrm{M}} Z_1 \hat{\mathrm{M}}^{-1} U^{\mathrm{T}},$$

where $Z_1 \in \Re^{p\times p}$, $Z_2 \in \Re^{(n-p)\times(n-p)}$ and the singular value decomposition of the matrix M is represented as $\mathrm{M} = U[\,\hat{\mathrm{M}} \quad 0\,]V^{\mathrm{T}}$ with the unitary matrices $U \in \Re^{p\times p}$, $V \in \Re^{n\times n}$

and a diagonal matrix $\hat{\mathrm{M}} \in \Re^{p\times p}$ *with positive diagonal elements in decreasing order.*

We firstly present a delay-dependent condition for the stability and mixed H_2/H_∞ performance of the second-order neutral linear system (1) with (3) for any time-varying delays satisfying (2) in the following theorem.

Theorem 1. *Under* (A1)*, for given scalars* $\gamma, h_M, d_M, r_M > 0, d_D < 1, h_D, r_D$*, the second-order neutral linear system* (1) *with any time-varying delays satisfying* (2) *is robustly stabilizable by* (3) *and satisfies* $J_2 \le J_0$ *under the constraint* $J_\infty < 0$*, if there exist some matrices* P_2*,* P_3*,* W*,* N_1*,* N_2*,* N_3*,* N_4 *and positive-definite matrices* P_1*,* Q_1*,* Q_2*,* Q_3 *and* H*, such that the matrix inequalities given in* Boxes I and II *are feasible.*

The terms in the matrix inequalities in Boxes I *and* II *are given by*

$$\begin{aligned}\Pi_{11} := &\ \mathrm{sym}\left\{\begin{bmatrix}\hat{I}^{\mathrm T}\tilde{I} & \tilde{I}^{\mathrm T}\\ \bar{A}_1 & M\end{bmatrix}^{\mathrm T} P + \begin{bmatrix}\tilde{I}^{\mathrm T}\\ 0\end{bmatrix}[0 \quad A_1^{\mathrm T}]HWP\right\}\\ &+r_M P^{\mathrm T}(W^{\mathrm T}H+I)H^{-1}(HW+I)P\\ &+r_M\begin{bmatrix}0\\ I\end{bmatrix}\begin{bmatrix}0\\ A_1\end{bmatrix}^{\mathrm T} H\begin{bmatrix}0\\ A_1\end{bmatrix}[0 \quad I]\\ &+\begin{bmatrix}Q_1+Q_2+\mathrm{sym}\{N_1+N_3\} & 0\\ 0 & Q_3\end{bmatrix}\\ &+(r_M+h_M)\begin{bmatrix}\tilde{I} & 0\\ 0 & I\end{bmatrix}^{\mathrm T} P_1\begin{bmatrix}\tilde{I} & 0\\ 0 & I\end{bmatrix},\end{aligned}$$

$$\Pi_{12} := P^{\mathrm T}\left(\begin{bmatrix}0\\ B_1\hat{I}\end{bmatrix} - W^{\mathrm T}H\begin{bmatrix}0\\ A_1\end{bmatrix}\tilde{I}\right)+\begin{bmatrix}N_2^{\mathrm T}-N_1\\ 0\end{bmatrix}$$

$$\Pi_{13} := P^{\mathrm T}\begin{bmatrix}0\\ -F_1KC\end{bmatrix}+\begin{bmatrix}N_4^{\mathrm T}-N_3\\ 0\end{bmatrix},$$

$$\Pi_{22} := -(1-r_D)Q_2-\mathrm{sym}\{N_2\},$$

$$\Pi_{33} := -(1-h_D)Q_1-\mathrm{sym}\{N_4\},$$

and

$$\bar{A}_1 := A\tilde{I}+(A_1+B)\hat{I}-FKC.$$

Moreover, an upper bound of the H_2 *performance measure is obtained by*

$$\begin{aligned}J_0 = &\ \xi(0)^{\mathrm T}P_1\xi(0)+\int_{-h(0)}^{0}\xi(s)^{\mathrm T}Q_1\xi(s)\mathrm{d}s\\ &+\int_{-r(0)}^{0}\xi(s)^{\mathrm T}Q_2\xi(s)\mathrm{d}s+\int_{-d(0)}^{0}\ddot{\phi}(s)^{\mathrm T}Q_3\ddot{\phi}(s)\mathrm{d}s\\ &+\int_{-r_M}^{0}\int_{\theta}^{0}\dot{\xi}(s)^{\mathrm T}P_1\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta\\ &+\int_{-h_M}^{0}\int_{\theta}^{0}\dot{\xi}(s)^{\mathrm T}P_1\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta\\ &+\int_{-r_M}^{0}(s+r_M)\ddot{\phi}(s)^{\mathrm T}\begin{bmatrix}0\\ A_1\end{bmatrix}^{\mathrm T}H\begin{bmatrix}0\\ A_1\end{bmatrix}\ddot{\phi}(s)\mathrm{d}s\end{aligned}\qquad(5)$$

where $\hat{I} := [I \quad 0]$, $\tilde{I} := [0 \quad I]$, $\xi(0) := \mathrm{col}\{\phi(0), \dot{\phi}(0)\}$ *and* $\xi(t) := \mathrm{col}\{\phi(t), \dot{\phi}(t)\}$ *for* $t \in [-\max\{h(0), r(0)\}, 0]$.

Proof. Firstly, we represent (1) in an equivalent descriptor model form as

$$\begin{cases}\ddot{x}(t)=\eta(t),\\ 0 = M\eta(t)+M_1\eta(t-d(t))+\bar{A}_1\xi(t)\\ \qquad -F_1KC\xi(t-h(t))+B_1\hat{I}\xi(t-r(t))\\ \qquad -A_1\int_{t-r(t)}^{t}\eta(s)\mathrm{d}s-Ew(t).\end{cases}\qquad(6)$$

Define the Lyapunov–Krasovskii functional

$$V(t)=\sum_{i=1}^{7}V_i(t),\qquad(7)$$

where

$$V_1(t)=\xi(t)^{\mathrm T}P_1\xi(t) := [\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}]TP\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix},$$

$$V_2(t)=\int_{t-h(t)}^{t}\xi(s)^{\mathrm T}Q_1\xi(s)\mathrm{d}s,$$

$$V_3(t)=\int_{t-r(t)}^{t}\xi(s)^{\mathrm T}Q_2\xi(s)\mathrm{d}s,$$

$$V_4(t)=\int_{t-d(t)}^{t}\eta(s)^{\mathrm T}Q_3\eta(s)\mathrm{d}s,$$

$$V_5(t)=\int_{-h_M}^{0}\int_{t+\theta}^{t}\dot{\xi}(s)^{\mathrm T}P_1\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta,$$

$$V_6(t)=\int_{-r_M}^{0}\int_{t+\theta}^{t}\dot{\xi}(s)^{\mathrm T}P_1\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta,$$

$$V_7(t)=\int_{t-r_M}^{t}(s-t+r_M)\eta(s)^{\mathrm T}\begin{bmatrix}0\\ A_1\end{bmatrix}^{\mathrm T}H\begin{bmatrix}0\\ A_1\end{bmatrix}\eta(s)\mathrm{d}s,$$

with $T=\mathrm{diag}\{I,0\}$ and $P=\begin{bmatrix}P_1 & 0\\ P_3 & P_2\end{bmatrix}$, where $P_1=P_1^{\mathrm T}>0$. Differentiating $V_1(t)$ along the system trajectory leads to the equation in Box III.

The term $\beta(t)$ in Box III is given by

$$\beta(t)=-2\int_{t-r(t)}^{t}[\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}]P^{\mathrm T}\begin{bmatrix}0\\ A_1\end{bmatrix}\eta(s)\mathrm{d}s.$$

Using inequality (4) in Lemma 1 for $a(s)=\mathrm{col}\{0,A_1\}\eta(s)$ and $b=P\,\mathrm{col}\{\xi(t),\eta(t)\}$ we obtain

$$\begin{aligned}\beta(t)\le &\ r_M[\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}]P^{\mathrm T}(W^{\mathrm T}H+I)H^{-1}(HW+I)P\\ &\times\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix}+2[\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}]P^{\mathrm T}W^{\mathrm T}H\begin{bmatrix}0\\ A_1\end{bmatrix}\\ &\times\tilde{I}(\xi(t)-\xi(t-r(t)))\\ &+\int_{t-r_M}^{t}\eta(s)^{\mathrm T}\begin{bmatrix}0\\ A_1\end{bmatrix}^{\mathrm T}H\begin{bmatrix}0\\ A_1\end{bmatrix}\eta(s)\mathrm{d}s.\end{aligned}\qquad(8)$$

Also, differentiating the second Lyapunov term in (7) gives

$$\begin{aligned}\dot{V}_2(t) &= \xi(t)^{\mathrm T}Q_1\xi(t)-(1-\dot{h}(t))\xi^{\mathrm T}(t-h(t))\\ &\quad\times Q_1\xi(t-h(t))\\ &\le \xi(t)^{\mathrm T}Q_1\xi(t)-(1-h_D)\xi^{\mathrm T}(t-h(t))\\ &\quad\times Q_1\xi(t-h(t)),\end{aligned}\qquad(9)$$

$$\Pi_1 = \begin{bmatrix} \Pi_{11} + \begin{bmatrix} (C_1\hat{I} + D_1KC)^{\mathrm{T}} \\ 0 \end{bmatrix} [\,(C_1\hat{I} + D_1KC) \quad 0\,] & \Pi_{12} + \begin{bmatrix} (C_1\hat{I} + D_1KC)^{\mathrm{T}} \\ 0 \end{bmatrix} C_2\hat{I} & \Pi_{13} + \begin{bmatrix} (C_1\hat{I} + D_1KC)^{\mathrm{T}} \\ 0 \end{bmatrix} D_2K\hat{I} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ M_1 \end{bmatrix} & -P^{\mathrm{T}}\begin{bmatrix} 0 \\ E \end{bmatrix} & \begin{bmatrix} r_MN_1 & h_MN_3 \\ 0 & 0 \end{bmatrix} \\ * & \Pi_{22} + \hat{I}^{\mathrm{T}}C_2^{\mathrm{T}}C_2\hat{I} & \hat{I}^{\mathrm{T}}C_2^{\mathrm{T}}D_2KC\hat{I} & 0 & 0 & \begin{bmatrix} r_MN_2 & 0 \end{bmatrix} \\ * & * & \Pi_{33} + \hat{I}^{\mathrm{T}}C^{\mathrm{T}}K^{\mathrm{T}}D_2^{\mathrm{T}}D_2KC\hat{I} & 0 & 0 & \begin{bmatrix} 0 & h_MN_4 \end{bmatrix} \\ * & * & * & -(1-d_D)Q_3 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & 0 \\ * & * & * & * & * & \begin{bmatrix} -r_MP_1 & 0 \\ * & -h_MP_1 \end{bmatrix} \end{bmatrix} < 0$$

Box I.

$$\Pi_2 = \begin{bmatrix} \Pi_{11} + \begin{bmatrix} S_1 + C^{\mathrm{T}}K^{\mathrm{T}}S_2KC & 0 \\ * & 0 \end{bmatrix} & \Pi_{12} & \Pi_{13} & P^{\mathrm{T}}\begin{bmatrix} 0 \\ M_1 \end{bmatrix} & \begin{bmatrix} r_MN_1 & h_MN_3 \\ 0 & 0 \end{bmatrix} \\ * & \Pi_{22} & 0 & 0 & \begin{bmatrix} r_MN_2 & 0 \end{bmatrix} \\ * & * & \Pi_{33} & 0 & \begin{bmatrix} 0 & h_MN_4 \end{bmatrix} \\ * & * & * & -(1-d_D)Q_3 & 0 \\ * & * & * & * & \begin{bmatrix} -r_MP_1 & 0 \\ * & -h_MP_1 \end{bmatrix} \end{bmatrix} < 0$$

Box II.

$$\begin{aligned} \dot{V}_1(t) &= 2\,\xi(t)^{\mathrm{T}}P_1\dot{\xi}(t) \\ &= 2[\,\xi(t)^{\mathrm{T}} \quad \eta(t)^{\mathrm{T}}\,]P^{\mathrm{T}}\begin{bmatrix} \dot{\xi}(t) \\ 0 \end{bmatrix} \\ &= 2[\,\xi(t)^{\mathrm{T}} \quad \eta(t)^{\mathrm{T}}\,]P^{\mathrm{T}} \\ &\quad \times \begin{bmatrix} \dot{\xi}(t) \\ M\,\eta(t) + M_1\,\eta(t-d(t)) + \bar{A}_1\,\xi(t) - F_1KC\,\xi(t-h(t)) + B_1\hat{I}\,\xi(t-r(t)) - A_1\displaystyle\int_{t-r(t)}^{t}\eta(s)\mathrm{d}s - Ew(t) \end{bmatrix} \\ &= 2[\,\xi(t)^{\mathrm{T}} \quad \eta(t)^{\mathrm{T}}\,]P^{\mathrm{T}}\left\{\begin{bmatrix} \hat{I}^{\mathrm{T}}\tilde{I} & \tilde{I}^{\mathrm{T}} \\ \bar{A}_1 & M \end{bmatrix}\begin{bmatrix} \xi(t) \\ \eta(t) \end{bmatrix} + \begin{bmatrix} 0 \\ M_1 \end{bmatrix}\eta(t-d(t)) + \begin{bmatrix} 0 \\ -F_1KC \end{bmatrix}\xi(t-h(t)) \right. \\ &\quad \left. + \begin{bmatrix} 0 \\ B_1\hat{I} \end{bmatrix}\xi(t-r(t)) - \begin{bmatrix} 0 \\ E \end{bmatrix}w(t)\right\} + \beta(t) \end{aligned}$$

Box III.

and, similarly,

$$\begin{aligned} \dot{V}_3(t) &= \xi(t)^{\mathrm{T}}Q_2\,\xi(t) - (1-\dot{r}(t))\xi^{\mathrm{T}}(t-r(t)) \\ &\quad \times Q_2\,\xi(t-r(t)) \\ &\le \xi(t)^{\mathrm{T}}Q_2\,\xi(t) - (1-r_D)\xi^{\mathrm{T}}(t-r(t)) \\ &\quad \times Q_2\,\xi(t-r(t)), \end{aligned} \tag{10}$$

$$\begin{aligned} \dot{V}_4(t) &= \eta(t)^{\mathrm{T}}Q_3\,\eta(t) - (1-\dot{d}(t))\eta^{\mathrm{T}}(t-d(t)) \\ &\quad \times Q_3\,\eta(t-d(t)) \\ &\le \eta(t)^{\mathrm{T}}Q_3\,\eta(t) - (1-d_D)\eta^{\mathrm{T}}(t-d(t)) \\ &\quad \times Q_3\,\eta(t-d(t)), \end{aligned} \tag{11}$$

$$\begin{aligned} \dot{V}_5(t) &= h_M\dot{\xi}(t)^{\mathrm{T}}P_1\dot{\xi}(t) - \int_{t-h_M}^{t}\dot{\xi}(s)^{\mathrm{T}}P_1\dot{\xi}(s)\mathrm{d}s \\ &\le h_M\dot{\xi}(t)^{\mathrm{T}}P_1\dot{\xi}(t) - \int_{t-h(t)}^{t}\dot{\xi}(s)^{\mathrm{T}}P_1\dot{\xi}(s)\mathrm{d}s \end{aligned} \tag{12}$$

$$\begin{aligned} \dot{V}_6(t) &= r_M\dot{\xi}(t)^{\mathrm{T}}P_1\dot{\xi}(t) - \int_{t-r_M}^{t}\dot{\xi}(s)^{\mathrm{T}}P_1\dot{\xi}(s)\mathrm{d}s \\ &\le r_M\dot{\xi}(t)^{\mathrm{T}}P_1\dot{\xi}(t) - \int_{t-r(t)}^{t}\dot{\xi}(s)^{\mathrm{T}}P_1\dot{\xi}(s)\mathrm{d}s \end{aligned} \tag{13}$$

and the time derivative of the last term of $V(t)$ in (7) is

$$\dot V_7(t) = r_M \eta(t)^{\mathrm T}\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm T} H \begin{bmatrix}0\\A_1\end{bmatrix}\eta(t) - \int_{t-r_M}^{t} \eta(s)^{\mathrm T}\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm T} H \begin{bmatrix}0\\A_1\end{bmatrix}\eta(s)\mathrm{d}s. \quad (14)$$

Moreover, from the Leibniz–Newton formula ($\xi(t) = \xi(t - r(t)) + \int_{t-r(t)}^{t} \dot\xi(s)\mathrm{d}s$), the following equation holds for any matrices N_1, N_2, N_3, N_4 with appropriate dimensions:

$$2(\xi(t)^{\mathrm T} N_1 + \xi^{\mathrm T}(t - r(t))N_2)(\xi(t) - \xi(t - r(t)) - \int_{t-r(t)}^{t} \dot\xi(s)\mathrm{d}s) = 0 \quad (15)$$

$$2(\xi(t)^{\mathrm T} N_3 + \xi^{\mathrm T}(t - h(t))N_4)(\xi(t) - \xi(t - h(t)) - \int_{t-h(t)}^{t} \dot\xi(s)\mathrm{d}s) = 0. \quad (16)$$

Using the obtained derivative terms given by Box III and (8)–(14), and adding the left sides of Eqs. (15) and (16) into the above, we obtain the following result for $\dot V(t)$

$$\begin{aligned}
\dot V(t) &= \sum_{i=1}^{7} \dot V_i(t) \\
&\le 2[\,\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}\,]P^{\mathrm T}\left\{\begin{bmatrix}\hat I^{\mathrm T}\tilde I & \hat I^{\mathrm T}\\ \bar A_1 & M\end{bmatrix}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} + \begin{bmatrix}0\\M_1\end{bmatrix}\eta(t-d(t)) + \begin{bmatrix}0\\-F_1KC\end{bmatrix}\xi(t-h(t)) \right.\\
&\quad \left. + \begin{bmatrix}0\\B_1\hat I\end{bmatrix}\xi(t-r(t)) - \begin{bmatrix}0\\E\end{bmatrix}w(t)\right\} \\
&\quad + r_M[\,\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}\,]P^{\mathrm T}(W^{\mathrm T}H + I)H^{-1}(HW + I) \\
&\quad \times P\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix} + 2[\,\xi(t)^{\mathrm T} \quad \eta(t)^{\mathrm T}\,]P^{\mathrm T}W^{\mathrm T}H\begin{bmatrix}0\\A_1\end{bmatrix} \\
&\quad \times \tilde I(\xi(t) - \xi(t-r(t))) + r_M\,\eta(t)^{\mathrm T}\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm T} H \\
&\quad \times \begin{bmatrix}0\\A_1\end{bmatrix}\eta(t) + \xi(t)^{\mathrm T}(Q_1 + Q_2 + \mathrm{sym}\{N_1 + N_3\})\,\xi(t) \\
&\quad - \xi^{\mathrm T}(t-h(t))((1-h_D)Q_1 + \mathrm{sym}\{N_4\})\xi(t-h(t)) \\
&\quad - \xi^{\mathrm T}(t-r(t))((1-r_D)Q_2 + \mathrm{sym}\{N_2\})\xi(t-r(t)) \\
&\quad + 2\,\xi^{\mathrm T}(t)(N_2^{\mathrm T} - N_1)\,\xi(t-r(t)) \\
&\quad + 2\,\xi^{\mathrm T}(t)(N_4^{\mathrm T} - N_3)\,\xi(t-h(t)) + \eta(t)^{\mathrm T}Q_3\,\eta(t) \\
&\quad - (1-d_D)\,\eta^{\mathrm T}(t-d(t))Q_3\eta(t-d(t)) \\
&\quad + (r_M + h_M)\dot\xi(t)^{\mathrm T}P_1\dot\xi(t) + r_M\vartheta^{\mathrm T}(t)\hat N_1P_1^{-1}\hat N_1^{\mathrm T}\vartheta(t) \\
&\quad + h_M\vartheta^{\mathrm T}(t)\hat N_2P_1^{-1}\hat N_2^{\mathrm T}\vartheta(t) \\
&\quad - \int_{t-r(t)}^{t}(\vartheta^{\mathrm T}(t)\hat N_1 + \dot\xi^{\mathrm T}(s)P_1)P_1^{-1}(\vartheta^{\mathrm T}(t)\hat N_1 \\
&\quad + \dot\xi^{\mathrm T}(s)P_1)^{\mathrm T}\mathrm{d}s - \int_{t-h(t)}^{t}(\vartheta^{\mathrm T}(t)\hat N_2 + \dot\xi^{\mathrm T}(s)P_1) \\
&\quad \times P_1^{-1}(\vartheta^{\mathrm T}(t)\hat N_2 + \dot\xi^{\mathrm T}(s)P_1)^{\mathrm T}\mathrm{d}s
\end{aligned} \quad (17)$$

where the vectors $\vartheta(t)$, $\hat N_1$ and $\hat N_2$ are, respectively, $\vartheta(t) := \mathrm{col}\{\xi(t), \eta(t), \xi(t - r(t)), \xi(t - h(t)), \eta(t - d(t)), w(t)\}$,

$\hat N_1 = \mathrm{col}\{N_1, 0, N_2, 0, 0, 0\}$ and

$\hat N_2 = \mathrm{col}\{N_3, 0, 0, N_4, 0, 0\}$.

The H_∞ performance measure in Definition 1 can be rewritten as

$$J_\infty = \int_0^\infty [z(t)^{\mathrm T}z(t) - \gamma^2 w(t)^{\mathrm T}w(t) + \dot V(t)]\mathrm{d}t. \quad (18)$$

Substituting the terms of

$$z(t) = (C_1\hat I + D_1KC)\,\xi(t) + C_2\hat I\,\xi(t - r(t)) + D_2KC\,\xi(t - h(t)),$$

$$\dot\xi(t) = \begin{bmatrix}\hat I^{\mathrm T}\tilde I & \hat I^{\mathrm T}\end{bmatrix}\begin{bmatrix}\xi(t)\\ \eta(t)\end{bmatrix},$$

and upper bound of $\dot V(t)$ in (17) results in (18) being less than the integrand $\vartheta(t)^{\mathrm T}\,\Pi_1\,\vartheta(t)$ where the matrix Π_1, by the Schur complement [56], is given in Box I. Now, if $\Pi_1 < 0$, then $J_\infty < 0$ which means that the L_2-gain from the disturbance $w(t)$ to the controlled output $z(t)$ is less than γ.

On the other hand, by applying the same Lyapunov–Krasovskii functional candidate (7) for the second-order neutral linear system (1), under $w(t) \equiv 0$, for the index J_2 in Definition 1 we get

$$\begin{aligned}
J_2 &\le \int_0^\infty [\xi^{\mathrm T}(t)S_1\,\xi(t) + \xi^{\mathrm T}(t)C^{\mathrm T}K^{\mathrm T}S_2KC\,\xi(t) + \dot V(t)]\mathrm{d}t \\
&\le \int_0^\infty \hat\vartheta^{\mathrm T}(t)\,\Pi_2\hat\vartheta(t)\mathrm{d}t
\end{aligned} \quad (19)$$

where $\hat\vartheta(t) := \mathrm{col}\{\xi(t), \eta(t), \xi(t - r(t)), \xi(t - h(t)), \eta(t - d(t))\}$ and the matrix Π_2 is given in Box II. Therefore, the condition $\Pi_2 < 0$ in (19) implies

$$\dot V(t) \le -\xi^{\mathrm T}(t)S_1\,\xi(t) - \xi^{\mathrm T}(t)C^{\mathrm T}K^{\mathrm T}S_2KC\,\xi(t) \quad (20)$$

or equivalently,

$$\int_0^\infty \dot V(t)\mathrm{d}t = \lim_{t\to\infty} V(t) - V(0) \le -\int_0^\infty [\xi^{\mathrm T}(t)S_1\,\xi(t) + \xi^{\mathrm T}(t)C^{\mathrm T}K^{\mathrm T}S_2KC\,\xi(t)]\mathrm{d}t. \quad (21)$$

Clearly, inequality (20) results in

$$\dot V(\phi(t), t) \le -\lambda_{\min}(S_1 + C^{\mathrm T}K^{\mathrm T}\,S_2KC)(\|\phi(t)\|^2 + \|\dot\phi(t)\|^2).$$

Note that $V(\phi(t), t) \ge \lambda_{\min}(P_1)\,(\|\phi(0)\|^2 + \|\dot\phi(0)\|^2)$. According to [27], using the Cauchy–Schwarz inequality, we have

$$\|\phi(\theta)\|^2 \le 2\,\|\phi(0)\|^2 - 2\theta\int_\theta^0 \|\dot\phi(u)\|^2\,\mathrm{d}u,$$

$$\int_{-h}^0 \|\dot\phi(\theta)\|^2\,\mathrm{d}\theta \le 2h\,\|\dot\phi(0)\|^2 + 2h^2\int_{-h}^0 \|\ddot\phi(u)\|^2\,\mathrm{d}u,$$

and

$$\int_{-h}^0 \|\phi(\theta)\|^2\,\mathrm{d}\theta \le 2h\,\|\phi(0)\|^2 + 4h^3\,\|\dot\phi(0)\|^2 + 4h^3\int_{-h}^0 \|\ddot\phi(u)\|^2\,\mathrm{d}u.$$

From this, after some manipulations, we obtain

$$V(\phi(t),t) \le V(\phi(0),0) \le \rho\left[\|\phi(0)\|^2 + \|\dot{\phi}(0)\|^2 + \int_{-\max\{h_M,d_M,r_M\}}^{0} \|\ddot{\phi}(\theta)\|^2 \,\mathrm{d}\theta\right]$$

where $\kappa := \max\{h_M, d_M, r_M\}$ and $\rho := \max(\rho_1, \rho_2, \rho_3)$ with

$$\rho_1 := \lambda_{\max}(P_1) + 2h_M\lambda_{\max}(Q_1) + 2r_M\lambda_{\max}(Q_2),$$
$$\rho_2 := 2(1+2r_M^2+2h_M^2)\lambda_{\max}(P_1) + 2h_M(1+2h_M^2) \times \lambda_{\max}(Q_1) + 2r_M(1+2r_M^2)\lambda_{\max}(Q_2)$$

and

$$\rho_3 := 2(r_M(1+2r_M^2) + h_M(1+2h_M^2))\lambda_{\max}(P_1) + 2h_M^2(1+2h_M^2)\lambda_{\max}(Q_1) + 2r_M^2(1+2r_M^2)\lambda_{\max}(Q_2) + \lambda_{\max}(Q_3) + 2r_M\lambda_{\max}\left(\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm{T}} H \begin{bmatrix}0\\A_1\end{bmatrix}\right).$$

Moreover, from (6) and the fact that $\xi(t)$ is square integrable on $[0,\infty)$, it follows that $D\eta_t \in L_2^n[0,\infty)$. Under (A1), the latter implies that $\eta(t-d(t)) \in L_2^n[0,\infty)$. Therefore, by Theorem 1.6 of Reference [47], we conclude that the neutral system (1) with $w(t) \equiv 0$ is asymptotically stabilizable by (3). Now, the H_2 performance measure for system (1) is established as

$$\int_0^\infty [\xi^{\mathrm{T}}(t)S_1\xi(t) + \xi^{\mathrm{T}}(t)C^{\mathrm{T}}K^{\mathrm{T}}S_2KC\xi(t)]\mathrm{d}t \le V(0) = J_0 \tag{22}$$

where J_0 is given by (5). This completes the proof. ■

Remark 4. It is easy to see that inequalities (5) imply $\Pi_{11} < 0$. Hence by Proposition 4.2 in Reference [18], the matrix P is non-singular. Then, according to the structure of the matrix P, the matrix $X := P^{-1}$ has the form

$$X = \begin{bmatrix} X_1 & 0 \\ X_3 & X_2 \end{bmatrix}, \tag{23}$$

where $X_i = P_i^{-1} (i = 1,2)$ and $X_3 = -X_2P_3X_1$.

Remark 5. According to the structure of matrix C, i.e., $C := \mathrm{diag}\{C_3, C_3\}$, with $\mathrm{rank}(C_3) = l < n$, Lemma 2 proposes that an equivalent condition on matrix equation $CX_1 = \hat{X}_1C$ is

$$X_1 = V\begin{bmatrix} X_{11} & 0 \\ 0 & X_{22}\end{bmatrix}V^{\mathrm{T}},$$
$$\hat{X}_1 = U\hat{C}X_{11}\hat{C}^{-1}U^{\mathrm{T}},$$

where $X_{11} \in \Re^{2l\times 2l}$, $X_{22} \in \Re^{2(n-l)\times 2(n-l)}$ and $C = U[\hat{C} \quad 0]V^{\mathrm{T}}$(the singular value decomposition of the matrix C), with $\mathrm{rank}(C) = 2l$, $U \in \Re^{2l\times 2l}$, $V \in \Re^{2n\times 2n}$ and $\hat{C} \in \Re^{2l\times 2l}$.

Now, we are in a position to give our main results on the existence of delay-dependent mixed H_2/H_∞ output-feedback control in the form of (3), and to show how to construct a desired control for the second-order neutral linear system in (1).

Theorem 2. *Consider the second-order neutral linear system (1) with time-varying delays (2). Under* (A1)*, for given scalars* $\gamma, h_M, d_M, r_M > 0, d_D < 1, h_D, r_D$*, there exists a delay-dependent mixed* H_2/H_∞ *output-feedback control in the form of (3) such that the resulting closed-loop system is robustly asymptotically stable and satisfies* $J_2 \le J_0$ *under the constraint* $J_\infty < 0$*, if there exist a scalar* α*, matrices* $\hat{N}_1, \hat{N}_2, \hat{N}_3, \hat{N}_4, \tilde{X}_1, X_2, X_3$ *and positive-definite matrices* $X_{11}, X_{22}, \hat{Q}_1, \hat{Q}_2, \bar{Q}_3$ *and* $\bar{H}$*, satisfying the LMIs in* Boxes IV and V.

The terms in Boxes IV and V are given by

$$\bar{\Pi}_{11} := \mathrm{sym}\{\hat{I}^{\mathrm{T}}\tilde{I}X_1 + \hat{N}_1 + \hat{N}_3 + \tilde{I}^{\mathrm{T}}X_3\} + \hat{Q}_1 + \hat{Q}_2,$$
$$\bar{\Pi}_{12} := X_1((A+\alpha A_1)\tilde{I} + (A_1+B)\hat{I})^{\mathrm{T}} - C^{\mathrm{T}}\tilde{X}_1^{\mathrm{T}}F^{\mathrm{T}} + \tilde{I}^{\mathrm{T}}X_2 + X_3^{\mathrm{T}}M^{\mathrm{T}}$$

and

$$\bar{\Pi}_{22} := \mathrm{sym}\{M X_2\}.$$

The desired control gain in (3) is given by

$$K = \tilde{X}_1\hat{X}_1^{-1} \quad \text{from LMIs in Boxes IV and V}, \tag{24}$$

and an upper bound of the H_2 *performance measure is obtained by*

$$J_0 = \xi(0)^{\mathrm{T}}X_1^{-1}\xi(0) + \int_{-h(0)}^{0}\xi(s)^{\mathrm{T}}X_1^{-1}\hat{Q}_1X_1^{-1}\xi(s)\mathrm{d}s + \int_{-r(0)}^{0}\xi(s)^{\mathrm{T}}X_1^{-1}\hat{Q}_2X_1^{-1}\xi(s)\mathrm{d}s + \int_{-d(0)}^{0}\ddot{\phi}(s)^{\mathrm{T}}\bar{Q}_3^{-1}\ddot{\phi}(s)\mathrm{d}s + \int_{-r_M}^{0}\int_{\theta}^{0}\dot{\xi}(s)^{\mathrm{T}}X_1^{-1}\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta + \int_{-h_M}^{0}\int_{\theta}^{0}\dot{\xi}(s)^{\mathrm{T}}X_1^{-1}\dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta + \int_{-r_M}^{0}(s+r_M)\ddot{\phi}(s)^{\mathrm{T}}\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm{T}}\bar{H}^{-1}\begin{bmatrix}0\\A_1\end{bmatrix}\ddot{\phi}(s)\mathrm{d}s \tag{25}$$

where the matrices X_1 *and* $\hat{X}_1$ *follow from* Remark 5.

Proof. By introducing $T := HWP$ as a new decision variable and applying the Schur complement to the matrix inequality in Box I in Theorem 1, we obtain the matrix in Box VI.

The term $\hat{\Pi}_{11}$ in Box VI is given by

$$\hat{\Pi}_{11} := \mathrm{sym}\left\{\begin{bmatrix}\hat{I}^{\mathrm{T}}\tilde{I} & \tilde{I}^{\mathrm{T}}\\ \bar{A}_1 & M\end{bmatrix}^{\mathrm{T}}P + \begin{bmatrix}\tilde{I}^{\mathrm{T}}\\0\end{bmatrix}[0 \quad A_1^{\mathrm{T}}]T\right\} + \begin{bmatrix}Q_1+Q_2+\mathrm{sym}\{N_1+N_3\} & 0\\ 0 & Q_3\end{bmatrix} + r_M\begin{bmatrix}0\\I\end{bmatrix}\begin{bmatrix}0\\A_1\end{bmatrix}^{\mathrm{T}}H\begin{bmatrix}0\\A_1\end{bmatrix}[0 \quad I].$$

Let

$$\zeta = \mathrm{diag}\{X^{\mathrm{T}}, X_1, X_1, \bar{Q}_3, I, X_1, X_1, X_1, I, \bar{H}\}$$

where $\bar{Q}_3 = Q_3^{-1}$ and $\bar{H} = H^{-1}$. Noting Remark 4 and premultiplying ζ and postmultiplying ζ^{T} to the matrix

$$
\begin{bmatrix}
\begin{bmatrix}\bar{\Pi}_{11} & \bar{\Pi}_{12}\\ * & \bar{\Pi}_{22}\end{bmatrix} & \begin{bmatrix}\hat{N}_2^{\mathrm{T}}-\hat{N}_1\\ (B_1\hat{I}-\alpha A_1\tilde{I})X_1\end{bmatrix} & \begin{bmatrix}\hat{N}_4^{\mathrm{T}}-\hat{N}_3\\ -F_1\tilde{X}_1C\end{bmatrix} & \begin{bmatrix}0\\ M_1\bar{Q}_3\end{bmatrix} & -\begin{bmatrix}0\\ E\end{bmatrix} & \begin{bmatrix}r_M\hat{N}_1\\ 0\end{bmatrix} & \begin{bmatrix}h_M\hat{N}_3\\ 0\end{bmatrix} & (r_M+h_M)\begin{bmatrix}X_1\hat{I}^{\mathrm{T}}\hat{I}+X_3^{\mathrm{T}}\tilde{I}\\ X_2^{\mathrm{T}}\tilde{I}\end{bmatrix} & \begin{bmatrix}X_1(C_1\hat{I})^{\mathrm{T}}+C^{\mathrm{T}}\tilde{X}_1^{\mathrm{T}}D_1^{\mathrm{T}}\\ 0\end{bmatrix} & r_M(\alpha+1)\bar{H} & \begin{bmatrix}X_3^{\mathrm{T}}\\ X_2^{\mathrm{T}}\end{bmatrix} & \begin{bmatrix}0 & X_3^{\mathrm{T}}A_1^{\mathrm{T}}\\ 0 & X_2^{\mathrm{T}}A_1^{\mathrm{T}}\end{bmatrix}\\
* & -(1-r_D)\hat{Q}_2-\mathrm{sym}\{\hat{N}_2\} & 0 & 0 & 0 & r_M\hat{N}_2 & 0 & 0 & X_1\hat{I}^{\mathrm{T}}C_2^{\mathrm{T}} & 0 & 0 & 0\\
* & * & -(1-h_D)\hat{Q}_1-\mathrm{sym}\{\hat{N}_4\} & 0 & 0 & 0 & h_M\hat{N}_4 & 0 & C^{\mathrm{T}}\tilde{X}_1^{\mathrm{T}}D_2^{\mathrm{T}} & 0 & 0 & 0\\
* & * & * & -(1-d_D)\bar{Q}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -r_M X_1 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -h_M X_1 & -(r_M+h_M)X_1 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -I & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -r_M\bar{H} & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -\bar{Q}_3 & 0\\
* & * & * & * & * & * & * & * & * & * & * & -r_M^{-1}\bar{H}
\end{bmatrix} < 0
$$

Box IV.

in Box VI and considering $TX = \alpha I$ to eliminate the nonlinearities in the matrix inequality we obtain the equation in Box VII (by the Schur complement).

The term $\tilde{\Pi}_{12}$ in Box VII is given by

$$\tilde{\Pi}_{12} := X_1(\bar{A}_1^{\mathrm{T}} + \alpha \tilde{I}^{\mathrm{T}} A_1^{\mathrm{T}}) + \tilde{I}^{\mathrm{T}} X_2 + X_3^{\mathrm{T}} M^{\mathrm{T}}$$

with

$$\hat{Q}_i = X_1^{\mathrm{T}} Q_i X_1 (i = 1, 2), \qquad \hat{N}_j = X_1^{\mathrm{T}} N_j X_1 \quad (j = 1, \dots, 4),$$

where the matrices $\hat{Q}_i, \hat{N}_j$ are, respectively, new decision variables instead of the matrices Q_i, N_j. Obviously, the matrix inequality in Box VII includes multiplication of control gain and the decision variable X_1. In the literature, much attention has been paid to the problems having this nature, namely bilinear matrix inequalities (BMIs) [57]. Now, by introducing $\tilde{X}_1 := K\hat{X}_1$ as a new decision variable instead of the matrix K, the obtained BMI in Box VII is converted into a convex programming problem written in terms of the matrix inequality in Box IV.

Similarly, by the Schur Complement the inequality $\Pi_2 < 0$ in Box II yields the equation in Box VIII.

Again, noting Remark 4 and applying the congruence transformation $\mathrm{diag}\{X^{\mathrm{T}}, X_1, X_1, \bar{Q}_3, X_1, X_1, X_1, I, \bar{H}\}$ to the matrix inequality in Box VIII, we readily obtain the matrix inequality in Box V. ∎

Remark 6. If $\mathrm{rank}(C_3) = l = n$, the matrix C is non-singular, it is clear that the matrix equation $CX_1 = \hat{X}_1 C$ is solvable on $\hat{X}_1$, i.e., $\hat{X}_1 = C X_1 C^{-1}$. In this case, the results of Theorem 3 are true for a full (non-diagonal) matrix X_1, i.e., $X_1 = \begin{bmatrix} X_{11} & X_{12} \\ * & X_{22} \end{bmatrix}$, and the desired control gain in (3) is given by $K = \tilde{X}_1 C X_1^{-1} C^{-1}$.

Remark 7. It is worth noting that in the case when $x(t) \in \Re^n$, $u(t) \in \Re^r$ and $y(t) \in \Re^l$. The number of the variables to be determined for the matrix inequalities in Boxes IV and V is $0.5\,n\,(51n + 23) + l(2l + 2r + 1) + (n - l)(2n - 2l + 1) + 2$.

Remark 8. We note that the problem of finding the smallest $\gamma > 0$, namely γ_0, is to determine whether the problem given by Boxes IV and V is feasible or not. It is called the feasibility problem. Also, the solutions of the problem can be found by solving the generalized eigenvalue problem in $\hat{N}_1$, $\hat{N}_2$, $\hat{N}_3$, $\hat{N}_4$, $\tilde{X}_1$, X_2, X_3, X_{11}, X_{22}, $\bar{H}$, $\hat{Q}_1$, $\hat{Q}_2$, $\bar{Q}_3$, α and $\lambda = \gamma^2$, which is a quasi-convex optimization problem as follows:

Minimize λ

subject to $X_{11} > 0, \quad X_{22} > 0, \quad \hat{Q}_1 > 0, \quad \hat{Q}_2 > 0, \quad \bar{Q}_3 > 0, \quad \bar{H} > 0, \quad \lambda > 0, \quad \alpha, \hat{N}_1, \hat{N}_2, \hat{N}_3, \hat{N}_4, \tilde{X}_1, X_2, X_3$, and LMIs in Boxes IV and V

$$
\left[\begin{array}{cccccccccccc}
\begin{bmatrix}\bar{\Pi}_{11} & \bar{\Pi}_{12}\\ * & \bar{\Pi}_{22}\end{bmatrix} & \begin{bmatrix}\hat{N}_2^{\mathrm{T}}-\hat{N}_1\\ (B_1\hat{I}-\alpha A_1\tilde{I})X_1\end{bmatrix} & \begin{bmatrix}\hat{N}_4^{\mathrm{T}}-\hat{N}_3\\ -F_1\tilde{X}_1C\end{bmatrix} & \begin{bmatrix}0\\ M_1\bar{Q}_3\end{bmatrix} & \begin{bmatrix}r_M\hat{N}_1\\ 0\end{bmatrix} & \begin{bmatrix}h_M\hat{N}_3\\ 0\end{bmatrix} & (r_M+h_M)\begin{bmatrix}X_1\tilde{I}^{\mathrm{T}}\hat{I}+X_3^{\mathrm{T}}\tilde{I}\\ X_2^{\mathrm{T}}\tilde{I}\end{bmatrix} & \begin{bmatrix}C^{\mathrm{T}}\tilde{X}_1^{\mathrm{T}}S_2\\ 0\end{bmatrix} & r_M(\alpha+1)\bar{H} & \begin{bmatrix}X_3^{\mathrm{T}}\\ X_2^{\mathrm{T}}\end{bmatrix} & \begin{bmatrix}0 & X_3^{\mathrm{T}}A_1^{\mathrm{T}}\\ 0 & X_2^{\mathrm{T}}A_1^{\mathrm{T}}\end{bmatrix} & \begin{bmatrix}X_1S_1\\ 0\end{bmatrix}\\
* & -(1-r_D)\hat{Q}_2-\mathrm{sym}\{\hat{N}_2\} & 0 & 0 & r_M\hat{N}_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & -(1-h_D)\hat{Q}_1-\mathrm{sym}\{\hat{N}_4\} & 0 & 0 & h_M\hat{N}_4 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & -(1-d_D)\bar{Q}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -r_MX_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & -h_MX_1 & 0 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & -(r_M+h_M)X_1 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & -S_2 & 0 & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & -r_M\bar{H} & 0 & 0 & 0\\
* & * & * & * & * & * & * & * & * & -\bar{Q}_3 & 0 & 0\\
* & * & * & * & * & * & * & * & * & * & -r_M^{-1}\bar{H} & 0\\
* & * & * & * & * & * & * & * & * & * & * & -S_1
\end{array}\right] < 0
$$

Box V.

$$
\left[\begin{array}{ccccccccc}
\hat{\Pi}_{11} & \Pi_{12} & \Pi_{13} & P^{\mathrm{T}}\begin{bmatrix}0\\ M_1\end{bmatrix} & -P^{\mathrm{T}}\begin{bmatrix}0\\ E\end{bmatrix} & \begin{bmatrix}r_MN_1 & h_MN_3\\ 0 & 0\end{bmatrix} & (r_M+h_M)\begin{bmatrix}\hat{I}^{\mathrm{T}}\hat{I}\\ \tilde{I}\end{bmatrix}P_1 & \begin{bmatrix}(C_1\hat{I}+D_1KC)^{\mathrm{T}}\\ 0\end{bmatrix} & r_M(T^{\mathrm{T}}+P^{\mathrm{T}})\\
* & \Pi_{22} & 0 & 0 & 0 & \begin{bmatrix}r_MN_2 & 0\end{bmatrix} & 0 & \hat{I}^{\mathrm{T}}C_2^{\mathrm{T}} & 0\\
* & * & \Pi_{33} & 0 & 0 & \begin{bmatrix}0 & h_MN_4\end{bmatrix} & 0 & \hat{I}^{\mathrm{T}}C^{\mathrm{T}}K^{\mathrm{T}}D_2^{\mathrm{T}} & 0\\
* & * & * & -(1-d_D)Q_3 & 0 & 0 & 0 & 0 & 0\\
* & * & * & * & -\gamma^2I & 0 & 0 & 0 & 0\\
* & * & * & * & * & \begin{bmatrix}-r_MP_1 & 0\\ * & -h_MP_1\end{bmatrix} & 0 & 0 & 0\\
* & * & * & * & * & * & -(r_M+h_M)P_1 & 0 & 0\\
* & * & * & * & * & * & * & -I & 0\\
* & * & * & * & * & * & * & * & -r_MH
\end{array}\right] < 0
$$

Box VI.

such that the bound γ_0 is given by $\gamma_0 = \sqrt{\lambda^*}$, where λ^* is the optimal value of the optimization problem. Note that a locally optimal point of a quasi-convex optimization problem with strictly quasi-convex objective is globally optimal. For the details, see Boyd et al. [56]. Various efficient convex optimization algorithms can be used to check whether the matrix inequalities in Boxes IV and V are feasible. In this paper, in order to solve the matrix inequalities in Boxes IV and V, we utilize Matlab's LMI Control Toolbox [58], which implements state-of-the-art interior-point algorithms, which are significantly faster than classical convex optimization algorithms [56].

Remark 9. From the process of the proofs of Theorems 1 and 2, it can be easily found that according to References [6,7], using the Leibniz–Newton formula and some free weighting matrices could help us to establish our test conditions with a less conservative approach and a relaxation on the bound of derivatives of the time-varying delays $r(t)$ and $h(t)$, i.e. $\dot{r}(t) \le r_D$ and $\dot{h}(t) \le h_D$, which mean that our method can deal with the systems with any fast time-varying delay case $r(t)$ and $h(t)$. Furthermore, the constraint $\dot{d}(t) \le d_D < 1$ in (2)(a) means that our method cannot handle completely fast time-varying neural delays $d(t)$ in (1) and it is a topic currently under study in the stabilization of second-order neutral systems.

Motivated by the idea of References [28–31], minimizing the upper bound of the H_2 performance measure is stated in the following theorem. Before proceeding further, we consider the

$$
\left[\begin{array}{cccccccccccc}
\begin{bmatrix} \bar{\Pi}_{11} & \tilde{\Pi}_{12} \\ * & \bar{\Pi}_{22} \end{bmatrix} & \begin{bmatrix} \hat{N}_2^{\mathrm{T}} - \hat{N}_1 \\ (B_1 \hat{I} - \alpha A_1 \tilde{I}) X_1 \end{bmatrix} & \begin{bmatrix} \hat{N}_4^{\mathrm{T}} - \hat{N}_3 \\ -F_1 K \hat{X}_1 C \end{bmatrix} & \begin{bmatrix} 0 \\ M_1 \end{bmatrix} \bar{Q}_3 & -\begin{bmatrix} 0 \\ E \end{bmatrix} & \begin{bmatrix} r_M \hat{N}_1 \\ 0 \end{bmatrix} & \begin{bmatrix} h_M \hat{N}_3 \\ 0 \end{bmatrix} & (r_M + h_M) \begin{bmatrix} X_1 \tilde{I}^{\mathrm{T}} \hat{I} + X_3^{\mathrm{T}} \tilde{I} \\ X_2^{\mathrm{T}} \tilde{I} \end{bmatrix} & \begin{bmatrix} X_1 (C_1 \hat{I})^{\mathrm{T}} + C^{\mathrm{T}} \hat{X}_1^{\mathrm{T}} K^{\mathrm{T}} D_1^{\mathrm{T}} \\ 0 \end{bmatrix} & r_M (\alpha + 1) \bar{H} & \begin{bmatrix} X_3^{\mathrm{T}} \\ X_2^{\mathrm{T}} \end{bmatrix} & \begin{bmatrix} 0 & X_3^{\mathrm{T}} A_1^{\mathrm{T}} \\ 0 & X_2^{\mathrm{T}} A_1^{\mathrm{T}} \end{bmatrix} \\
* & -(1 - r_D) \hat{Q}_2 - \mathrm{sym}\{\hat{N}_2\} & 0 & 0 & 0 & r_M \hat{N}_2 & 0 & 0 & X_1 \hat{I}^{\mathrm{T}} C_2^{\mathrm{T}} & 0 & 0 & 0 \\
* & * & -(1 - h_D) \hat{Q}_1 - \mathrm{sym}\{\hat{N}_4\} & 0 & 0 & 0 & h_M \hat{N}_4 & 0 & C^{\mathrm{T}} \hat{X}_1^{\mathrm{T}} K^{\mathrm{T}} D_2^{\mathrm{T}} & 0 & 0 & 0 \\
* & * & * & -(1 - d_D) \bar{Q}_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -\gamma^2 I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -r_M X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & -h_M X_1 & -(r_M + h_M) X_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & -I & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & 0 & 0 & 0 \\
* & * & * & * & * & * & * & * & * & -r_M \bar{H} & 0 & 0 \\
* & * & * & * & * & * & * & * & * & * & -\bar{Q}_3 & 0 \\
* & * & * & * & * & * & * & * & * & * & * & -r_M^{-1} \bar{H}
\end{array}\right] < 0
$$

Box VII.

$$
\begin{bmatrix}
\hat{\Pi}_{11} + \begin{bmatrix} S_1 & 0 \\ * & 0 \end{bmatrix} & \Pi_{12} & \Pi_{13} & P^{\mathrm{T}} \begin{bmatrix} 0 \\ M_1 \end{bmatrix} & \begin{bmatrix} r_M N_1 \\ 0 \end{bmatrix} & \begin{bmatrix} h_M N_3 \\ 0 \end{bmatrix} & r_M \begin{bmatrix} \tilde{I}^{\mathrm{T}} \hat{I} \\ \tilde{I} \end{bmatrix} P_1 & \begin{bmatrix} C^{\mathrm{T}} K^{\mathrm{T}} \\ 0 \end{bmatrix} S_2 & r_M (T^{\mathrm{T}} + P^{\mathrm{T}}) \\
* & \Pi_{22} & 0 & 0 & r_M N_2 & 0 & 0 & 0 & 0 \\
* & * & \Pi_{33} & 0 & 0 & h_M N_4 & 0 & 0 & 0 \\
* & * & * & -(1 - d_D) Q_3 & 0 & 0 & 0 & 0 & 0 \\
* & * & * & * & -r_M P_1 & 0 & 0 & 0 & 0 \\
* & * & * & * & * & -h_M P_1 & 0 & 0 & 0 \\
* & * & * & * & * & * & -r_M P_1 & 0 & 0 \\
* & * & * & * & * & * & * & -S_2 & 0 \\
* & * & * & * & * & * & * & * & -r_M H
\end{bmatrix} < 0
$$

Box VIII.

following notations:

$$\int_{-h(0)}^{0} \xi(s)\,\xi(s)^{\mathrm{T}} \mathrm{d}s := \aleph_1 \aleph_1^{\mathrm{T}},$$

$$\int_{-r(0)}^{0} \xi(s)\,\xi(s)^{\mathrm{T}}\, \mathrm{d}s := \aleph_2 \aleph_2^{\mathrm{T}},$$

$$\int_{-d(0)}^{0} \ddot{\phi}(s) \ddot{\phi}(s)^{\mathrm{T}}\, \mathrm{d}s := \aleph_3 \aleph_3^{\mathrm{T}},$$

$$\int_{-r_M}^{0} (s + r(0)) \dot{\xi}(s) \dot{\xi}(s)^{\mathrm{T}} \mathrm{d}s := \aleph_4 \aleph_4^{\mathrm{T}},$$

$$\int_{-h_M}^{0} (s + h(0)) \dot{\xi}(s) \dot{\xi}(s)^{\mathrm{T}} \mathrm{d}s := \aleph_5 \aleph_5^{\mathrm{T}}, \quad \text{and}$$

$$\int_{-r_M}^{0} (s + r_M) \ddot{\phi}(s) \ddot{\phi}(s)^{\mathrm{T}} \mathrm{d}s := \aleph_6 \aleph_6^{\mathrm{T}}.$$

Theorem 3. *Consider the following convex optimization problem*

$$\mathrm{Min} \left\{ \alpha + \sum_{i=1}^{6} \mathrm{tr}(Z_i) \right\} \tag{26}$$

subject to (i) *LMIs in* Boxes IV and V

(ii) $\begin{bmatrix} -\alpha & \xi(0)^{\mathrm{T}} \\ * & -X_1 \end{bmatrix} < 0$, (iii) $\begin{bmatrix} -Z_1 & \aleph_1^{\mathrm{T}} \\ * & -X_1 \end{bmatrix} < 0$,

(iv) $\begin{bmatrix} -Z_2 & \aleph_2^{\mathrm{T}} \\ * & -X_2 \end{bmatrix} < 0$, (v) $\begin{bmatrix} -Z_3 & \aleph_3^{\mathrm{T}} \\ * & -\bar{Q}_3 \end{bmatrix} < 0$,

(vi) $\begin{bmatrix} -Z_4 & \aleph_4^{\mathrm{T}} \\ * & -X_1 \end{bmatrix} < 0$, (vii) $\begin{bmatrix} -Z_5 & \aleph_5^{\mathrm{T}} \\ * & -X_1 \end{bmatrix} < 0$,

(viii) $\begin{bmatrix} -Z_6 & \aleph_6^{\mathrm{T}} \begin{bmatrix} 0 \\ A_1 \end{bmatrix}^{\mathrm{T}} \\ * & -\bar{H} \end{bmatrix} < 0.$

For the second-order neutral linear system (1)–(2) with given scalars $\gamma, h_M, d_M, r_M > 0$, $d_D < 1, r_D$ and h_D, if the optimization problem (26) has a scalar α, positive-definite matrices X_{11}, X_{22}, $\bar{Q}_3$, $\bar{H}$, $\{Z_i\}_{i=1}^{6}$, and matrices $\{\hat{N}_i\}_{i=1}^{4}$, $\tilde{X}_1$, X_2, X_3, then the control law (3) is a suboptimal delay-dependent mixed H_2/H_∞ output-feedback control which ensures the minimization of the upper bound of J_2 under the constraint $J_\infty < 0$.

Proof. According to Theorem 2, (i) in (26) is clear. Also, by applying the Schur Complement, it is easy to see that (ii) is equivalent to

$$-\alpha + \xi(0)^{\mathrm{T}} X_1^{-1} \xi(0) < 0. \tag{27}$$

By considering $Q_i = X_1^{-1}$ (or $\hat{Q}_i = X_1$) for $i = 1, 2$, to remove the present nonlinearities in the optimization technique, the second term on the right-hand side in (33) can be rewritten as

$$\begin{aligned} &\int_{-h(0)}^{0} \xi(s)^{\mathrm{T}} X_1^{-1} \hat{Q}_1 X_1^{-1} \xi(s)\mathrm{d}s \\ &= \int_{-h(0)}^{0} [\mathrm{tr}(\xi(s)^{\mathrm{T}} Q_1\, \xi(s))]\mathrm{d}s \\ &= \mathrm{tr}(\aleph_1^{\mathrm{T}} X_1^{-1} \aleph_1) < \mathrm{tr}(Z_1). \end{aligned} \tag{28}$$

Therefore, we get

$$\aleph_1^{\mathrm{T}} X_1^{-1} \aleph_1 < Z_1 \tag{29}$$

and by applying the Schur Complement, the LMI (iii) is easily obtained. Similarly, the third term on the right-hand side in (33) results in the LMI (iv). Furthermore,

$$\begin{aligned} &\int_{-d(0)}^{0} \ddot{\phi}(s)^{\mathrm{T}}\, \bar{Q}_3^{-1} \ddot{\phi}(s)\mathrm{d}s = \int_{-d(0)}^{0} [\mathrm{tr}(\ddot{\phi}(s)^{\mathrm{T}}\, \bar{Q}_3^{-1} \ddot{\phi}(s))]\mathrm{d}s \\ &= \mathrm{tr}(\aleph_3^{\mathrm{T}}\, \bar{Q}_3^{-1} \aleph_3) < \mathrm{tr}(Z_3) \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\int_{-r_M}^{0} \int_{\theta}^{0} \dot{\xi}(s)^{\mathrm{T}} X_1^{-1} \dot{\xi}(s)\mathrm{d}s\mathrm{d}\theta \\ &= \int_{-r_M}^{0} [\mathrm{tr}((s + r(0))\dot{\xi}(s)^{\mathrm{T}} X_1^{-1} \dot{\xi}(s))]\mathrm{d}s \\ &= \mathrm{tr}(\aleph_4^{\mathrm{T}} X_1^{-1} \aleph_4) < \mathrm{tr}(Z_4) \end{aligned} \tag{31}$$

and by applying the Schur Complement, the LMIs (v) and (vi) are concluded. Similarly, the last two terms on the right-hand side in (33) result in the LMIs (vii) and (viii). Therefore, it follows that

$$J_0 < \alpha + \sum_{i=1}^{6} \mathrm{tr}(Z_i). \tag{32}$$

Hence, if there exists a solution set to LMIs (26), the suboptimal mixed H_2/H_∞ output-feedback control $u(t) = \tilde{X}_1 \hat{X}_1^{-1} C\, \xi(t)$ minimizes the upper bound of the H_2 performance measure J_2 of the resulting closed-loop system (1) with (3). ■

4. Numerical examples

To illustrate the effectiveness of the approach, two numerical examples are presented. All the numerical examples were carried out with Matlab LMI Control Toolbox [58] running on a 2.67 GHz Pentium processor with 512 MB RAM.

Example 1. Consider the second-order nominal system of neutral type with constant delay $0 < r \le r_M$ and $|\, c\, | < 1$,

$$\begin{cases} \ddot{x}(t) - \begin{bmatrix} c & 0 \\ 1 & c \end{bmatrix} \ddot{x}(t-r) - \begin{bmatrix} c-1 & 0 \\ -1 & c-1 \end{bmatrix} \dot{x}(t-r) \\ \quad + \begin{bmatrix} 3 & 0 \\ 0 & 1.9 \end{bmatrix} \dot{x}(t) + \begin{bmatrix} 2 & 0 \\ 0 & 0.9 \end{bmatrix} x(t) \\ \quad + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x(t - r(t)) = 0, & \text{(a)} \\ x(t) = 0, \quad t \in [-\max\{r_M\}, 0]\,, & \text{(b)} \\ \dot{x}(t) = 0, \quad t \in [-\max\{r_M\}, 0]\,. & \text{(c)} \end{cases} \tag{33}$$

It is seen in [11] that the exact stability limit can be analytically calculated as follows:

(i) For $|c| < 1$ and $c \neq 0$,

$$r_M^{\text{analytical}} = \frac{1}{\omega} \arccos\left(\frac{c\omega^2 - 0.9}{1 + c^2\omega^2}\right) \tag{34}$$

where $\omega = \sqrt{\frac{-1+1.19\,c^2+\sqrt{1-1.62\,c^2+0.6561\,c^4}}{2(c^2-c^4)}}$.

(ii) For $c = 0$, $r_M^{\text{analytical}} = 6.17258$.

In order to show the improvement developed in this paper, some comparisons for the upper bounds of time delay to guarantee the asymptotic stability are shown in Table 1. The effect of parameter c on the maximum time delay for stability r_M is also shown in Table 1. We can conclude that the obtained results in this paper are less conservative than the existing results in References [9,10].

Example 2. Consider the undamped mass–spring system which is the nominal delay-free model taken from [38], where system matrices are given by

$$M = 10 I_3, \quad M_1 = 3 I_3, \quad A = A_1 = 0_3,$$

$$B = \begin{bmatrix} 40 & -40 & 0 \\ -40 & 80 & -40 \\ 0 & -40 & 80 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 4 & -4 & 0 \\ -4 & 8 & -4 \\ 0 & -4 & 8 \end{bmatrix},$$

$$F = \begin{bmatrix} 1 & 2 \\ 3 & 2 \\ 3 & 4 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0.1 & 0.2 \\ 0.3 & 0.2 \\ 0.3 & 0.4 \end{bmatrix},$$

Table 1
Some comparisons for the upper bounds of time delay r

c	0	0.1	0.3	0.5	0.7	0.9
Results of [10]	4.35	4.33	4.10	3.62	2.73	0.99
Results of [9]	4.47	4.42	4.17	3.69	2.87	1.41
Results of this paper	6.05	5.90	5.38	4.50	3.25	1.44
$r_M^{\text{analytical}}$	6.1725	6.0372	5.5491	4.7388	3.5092	1.5708

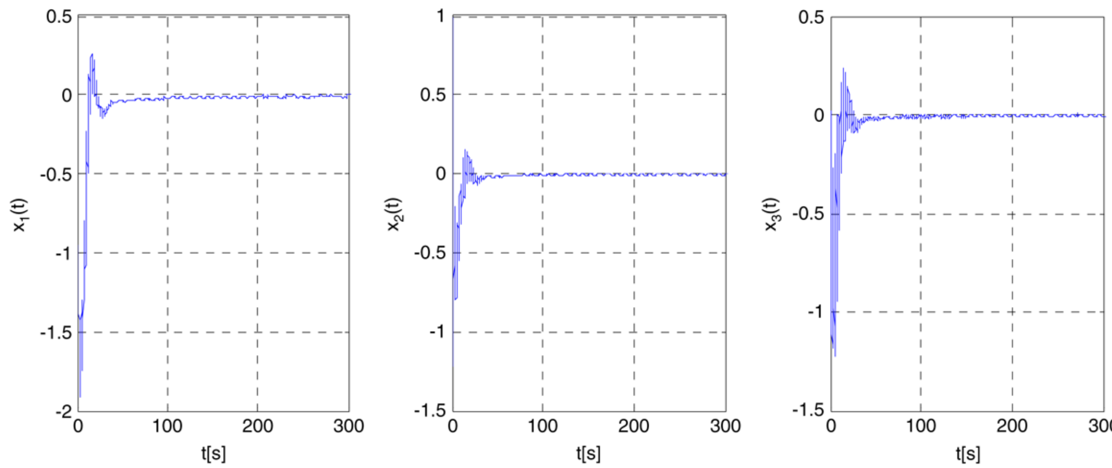

Fig. 1. Curves of state response of the system.

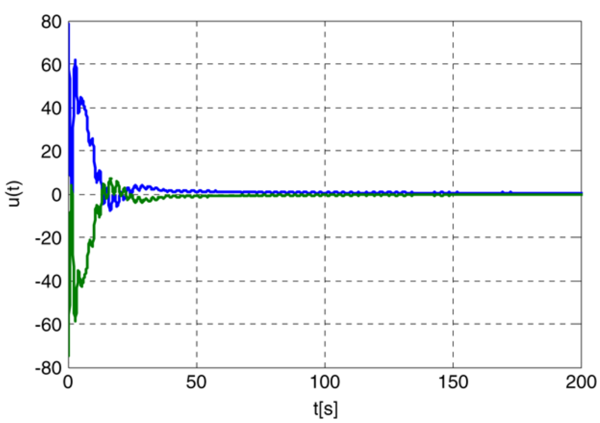

Fig. 2. Control law for the system.

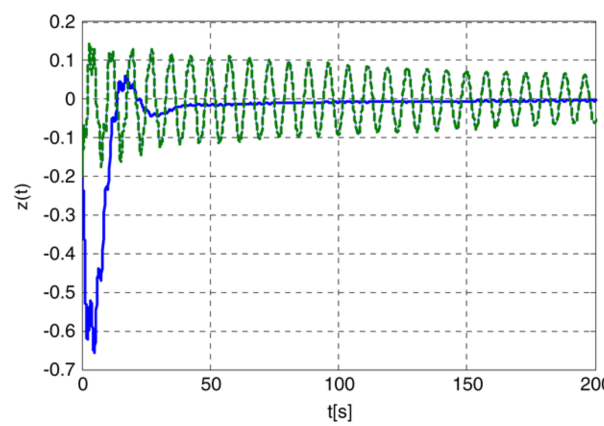

Fig. 3. Response of the controlled output: (a) open-loop system (dashed line) and (b) closed-loop system (solid line).

$$C_1 = C_2 = \begin{bmatrix} 0.1 \\ 0.1 \\ 0.1 \end{bmatrix}^{\mathrm{T}}, \quad C_3 = \begin{bmatrix} 0.1 \\ -0.1 \\ 0.1 \end{bmatrix}^{\mathrm{T}}, \quad D_1 = D_2 = 0,$$

with $S_1 = I_6$, $S_2 = I_2$, constant neutral delay $d = 0.1$ and time-varying delays $r(t) = h(t) = 0.25\sin^2(2\pi t)$ where $r_M = h_M = 0.25$ and $r_D = h_D = 1.5708$. For simulation purpose, an exogenous disturbance input is set as

$$w(t) = \frac{1}{2 + 0.5\,t}, \quad t \geq 0 \tag{35}$$

which belongs to $[0, \infty)$ and is imposed on the system.

It is required to design a robust mixed H_2/H_∞ output-feedback control law (3) such that the closed-loop system is asymptotically stable and satisfies mixed H_2/H_∞ performance measure. To this end, in light of Theorem 2 and Remark 8, we solved the LMIs in Boxes IV and V and obtained the minimum value of the parameter γ in optimal H_∞ performance measure as 0.9250. Moreover, according to Theorem 2, the control gain is given by

$$K = \begin{bmatrix} -248.6728 & 37.5694 \\ 236.5963 & -38.2491 \end{bmatrix}.$$

For initial conditions $x(0) = (-1, 1, -1)$, $\dot{x}(0) = (0.1, 0.1, 0.1)$, the simulation results are shown in Figs. 1–3 and the corresponding suboptimal H_2 performance measure of the closed-loop system is $J_0 = 17.3925$. The state trajectories of the system are plotted in Fig. 1. The curve of output-feedback control is also shown in Fig. 2. To observe the H_∞ performance, the response of the controlled output, i.e., $z(t)$, is depicted and compared with the output signal in the open-loop system under the disturbance (35). It is seen from Fig. 3 that the closed-loop system is asymptotically stable and the mixed H_2/H_∞ output-feedback controller (3) reduces the effect of the disturbance input $w(t)$ on the controlled output.

5. Conclusion

A mixed H_2/H_∞ output-feedback control design method has been presented in this paper for second-order neutral linear systems with time-varying state and input delays. Delay-dependent sufficient conditions for the design of a desired control were given by the Lyapunov–Krasovskii method in terms of LMIs. A controller guaranteeing asymptotic stability, and a mixed H_2/H_∞ performance of the closed-loop system was developed directly instead of coupling the model to a first-order neutral system using some free weighting matrices. Two numerical examples have been given to show the effectiveness of the method. Finally, it is worth commenting that the results of this work can be further extended to neutral systems with multiple time-varying delays.

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