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ORIGINAL ARTICLE

An ISS small gain theorem for general networks

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Abstract We provide a generalized version of the *nonlinear small gain theorem* for the case of more than two coupled input-to-state stable systems. For this result the interconnection gains are described in a nonlinear gain matrix, and the small gain condition requires bounds on the image of this gain matrix. The condition may be interpreted as a nonlinear generalization of the requirement that the spectral radius of the gain matrix is less than 1. We give some interpretations of the condition in special cases covering two subsystems, linear gains, linear systems and an associated lower-dimensional discrete time dynamical system.

Keywords Interconnected systems · Input-to-state stability · Small gain theorem · Large-scale systems · Monotone maps

1 Introduction

Stability is one of the fundamental concepts in the analysis and design of nonlinear dynamical systems. In particular, the notions of input-to-state stability (ISS) and nonlinear gains have proved to be efficient tools for the qualitative description of stability of nonlinear control systems. There are different equivalent formulations of ISS: In

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terms of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions (see below), via Lyapunov functions, as an asymptotic stability property combined with asymptotic gains, and others, see [27]. A more quantitative but equivalent formulation, which captures the long-term dynamic behavior of the system, is the notion of input-to-state dynamical stability (ISDS), see [9].

One of the interesting properties in the study of ISS systems is that under certain conditions ISS is preserved if these systems are connected in cascades or feedback loops. In this paper we generalize the existing results in this area. In particular, we obtain a general condition that guarantees ISS of a general system described as an interconnection of several ISS subsystems.

The earliest interconnection result on ISS systems states that cascades of ISS systems are again ISS, see e.g., [24–26]. Furthermore, small gain theorems for the case of two ISS systems in a feedback interconnection have been obtained in [9, 13, 14]. These results state in one way or another that if the composition of the gain functions of ISS subsystems is smaller than the identity, then the overall system is ISS.

The papers [9, 13, 14] use different approaches to the formulation of small gain conditions that yield sufficient stability criteria: In [13] the proof is trajectory-based and uses properties of $\mathcal{KL}$ and $\mathcal{K}_\infty$ functions. This approach requires that the composition of the gains is smaller than the identity in a robust sense, see below for the precise statement. We show in Example 18 that within the context of the summation formulation of ISS the robustness condition cannot be weakened. The ISS result of [13] will turn out to be a special case of our main result. That paper also covers practical ISS results, which we do not treat here. An ISS-Lyapunov function for the feedback system is constructed in [14] as some combination of the corresponding ISS-Lyapunov functions of both subsystems. The key assumption of the proof in that paper is that the gains are provided in terms of the max-formulation of ISS, by which the authors need not resort to a robust version of the small gain condition. The proof of the small gain theorem in [9] is based on the ISDS property, and conditions for asymptotic stability of the feedback loop without inputs are derived.

General stability conditions for large-scale interconnected systems have been obtained by various authors in other contexts. In [20] sufficient conditions for the asymptotic stability of a composite system are stated in terms of the negative definiteness of some test matrix. This matrix is defined through the given Lyapunov functions of the interconnected subsystems. Similarly, in [22] conditions for the stability of interconnected systems in terms of Lyapunov functions of the individual systems are obtained.

In [23] Šiljak considers structural perturbations and their effects on the stability of composite systems using Lyapunov theory. The method is to reduce each subsystem to a one-dimensional one, such that the stability properties of the reduced representation imply the same stability properties of the original interconnected system. In some cases the lower dimensional representation gives rise to an interconnection matrix $\bar{W}$, such that quasi-dominance or negative definiteness of $\bar{W}$ yield asymptotic stability of the composite system.

In [32] small gain type theorems for general interconnected systems with linear gains can be found. These results are of the form that the spectral radius of a gain matrix should be less than 1 to conclude stability. The result obtained here may be regarded as a nonlinear generalization in the same spirit.

In [29] Teel gives a small gain theorem for systems with outputs satisfying an ISS-related stability property and having saturated interconnections. Here, it is also shown how this result may be used in a large variety of specific control applications. In [30] Razumikhin-type theorems for functional differential equations are derived, which are based on the ISS small gain theorems in [13] and [29].

In this paper we consider a system that consists of two or more ISS subsystems. We provide conditions by which the stability question of the overall system can be reduced to consideration of stability of the subsystems. We choose an approach using asymptotic gains and global stability to prove the ISS stability result for general interconnected systems. The generalized small gain condition we obtain is, that for some monotone operator $\tilde{\Gamma}$ related to the gains of the individual systems the condition

$$\tilde{\Gamma}(s) \not\geq s \tag{1}$$

holds for all $s \geq 0$, $s \neq 0$ (in the sense of the component-wise ordering of the positive orthant). We discuss interpretations of this condition in Sect. 5.

In [4–7] the authors prove that condition (1) is also sufficient for the existence of an ISS Lyapunov function for interconnections of ISS subsystems. An explicit construction of this Lyapunov function in terms of the Lyapunov functions of the subsystems is described and a numerical algorithm for the construction is provided.

While the general problem of establishing ISS for networks of ISS systems can be approached by repeated application of the cascade property and the known small gain theorem, in general this can be cumbersome and it is by no means obvious in which order subsystems have to be chosen to proceed in such an iterative manner. Hence an extension of the known small gain theorem to larger interconnections is needed. In this paper we obtain this extension for the general case.

The paper is organized as follows. In Sect. 2 notation and necessary concepts are introduced. The problem is stated in Sect. 3. In Sect. 4 we prove the main result, which generalizes the known small gain theorems, and consider the special case of linear gains. In Sect. 5 the small gain condition of the main result is discussed and we show in which way it may be interpreted as an extension of the linear condition that the spectral radius of the gain matrix has to be less than 1. It is shown that the small gain condition is intimately related to the stability of a discrete time monotone dynamical system defined using the gain matrix. Section 6 concludes this work. In the Appendix we recall some relations between non-negative matrices and directed graphs.

2 Preliminaries

Notation By x^{T} we denote the transpose of a vector $x \in \mathbb{R}^n$. In the following we denote $\mathbb{R}_+ := [0, \infty)$. For $x, y \in \mathbb{R}^n$, we use the standard partial order induced by the positive orthant. It is given by

$$x \geq y \;\Leftrightarrow\; x_i \geq y_i,\; i = 1, \ldots, n, \quad \text{and} \quad x > y \;\Leftrightarrow\; x_i > y_i, \quad i = 1, \ldots, n. \tag{2}$$

By $\mathbb{R}^n_+$ we denote the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$.

For nonempty index sets $I, J \subset \{1, \dots, n\}$ we denote by $\#I$ the number of elements of I and by y_I the restriction

$$y_I := (y_i)_{i \in I}$$

for vectors $y \in \mathbb{R}^n_+$. Let P_I denote the projection of $\mathbb{R}^n_+$ onto $\mathbb{R}^{\#I}_+$ mapping y to y_I, and R_J the anti-projection $\mathbb{R}^{\#J}_+ \to \mathbb{R}^n_+$, defined by

$$x \mapsto \sum_{k=1}^{\#J} x_k e_{j_k},$$

where $\{e_k\}_{k=1,\dots,n}$ denotes the standard basis of $\mathbb{R}^n$ and J is

$$J = \{j_1, \dots, j_{\#J}\}, \quad \text{with} \quad j_k < j_{k+1}, \;\; k = 1, \dots, \#J - 1.$$

This allows to define the restriction $A_{IJ} : \mathbb{R}^{\#J}_+ \to \mathbb{R}^{\#I}_+$ of a mapping $A : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $A_{IJ} = P_I \circ A \circ R_J$.

For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1 (i) A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$. It is of class $\mathcal{K}_\infty$ if, in addition, it is unbounded.

(ii) A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$ if, for each fixed t, the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $\beta(s, \cdot)$ is non-increasing and tends to zero for $t \to \infty$.

Let $|\cdot|$ denote some norm in $\mathbb{R}^n$, and let in particular $|x|_{\max} = \max_i |x_i|$ be the maximum norm. The essential supremum norm on essentially bounded functions defined on $\mathbb{R}_+$ is denoted by $\|\cdot\|_\infty$.

Stability concepts Consider a system

$$\dot{x} = f(x, u), \quad x \in \mathbb{R}^N, \quad u \in \mathbb{R}^M, \tag{3}$$

such that for all initial values ξ and all essentially bounded measurable inputs u unique solutions exist for all positive times. We denote these solutions by $x(\cdot; \xi, u)$.

Definition 2 (input-to-state stability) The system (3) is called *input to state stable* (ISS), if there exist functions β of class $\mathcal{KL}$ and γ of class $\mathcal{K}$, such that the inequality

$$|x(t; \xi, u)| \leq \beta(|\xi|, t) + \gamma(||u||_\infty)$$

holds for all $t \geq 0$, $\xi \in \mathbb{R}^N$, $u : \mathbb{R}_+ \to \mathbb{R}^M$ measurable and essentially bounded.

Definition 3 (asymptotic gain) System (3) has the *asymptotic gain* (AG) property, if there exists $\gamma \in \mathcal{K}_\infty$, such that for all initial states $\xi \in \mathbb{R}^N$ and all $u(\cdot) \in L^\infty([0,\infty);\mathbb{R}^M)$ the estimate

$$\limsup_{t\to\infty} |x(t,\xi,u)| \leq \gamma(||u||_\infty) \tag{4}$$

holds.

The asymptotic gain property states, that every trajectory must ultimately stay not far from zero, depending on the magnitude of $\|u\|_\infty$.

Note that (4) is equivalent to

$$\limsup_{t\to\infty}|x(t,\xi,u)| \leq \gamma(\text{ess.}\ \limsup\nolimits_{t\to\infty}|u(t)|) = \text{ess.}\ \limsup\nolimits_{t\to\infty}\gamma(|u(t)|), \tag{5}$$

see [27].

Definition 4 (global stability) System (3) has the *global stability* (GS) property, if there exist $\sigma_1, \sigma_2 \in \mathcal{K}_\infty$, such that for all initial states $\xi \in \mathbb{R}^N$, all $t \geq 0$, and all $u(\cdot) \in L^\infty([0,\infty);\mathbb{R}^M)$ the estimate

$$|x(t,\xi,u)| \leq \sigma_1(|\xi|) + \sigma_2(||u||_\infty) \tag{6}$$

holds.

Definition 5 The system (3) is said to be *globally asymptotically stable at zero* (0-GAS), if there exists a $\beta \in \mathcal{KL}$, such that for all initial conditions $\xi \in \mathbb{R}^N$

$$|x(t;\xi,0)| \leq \beta(|\xi|,t). \tag{7}$$

Thus 0-GAS holds, if, when the input u is set to zero, the system (3) is globally asymptotically stable at $x^* = 0$.

By a result of Sontag and Wang [27] the asymptotic gain property and global asymptotic stability at 0 together are equivalent to ISS.

3 System interconnection

Consider n interconnected control systems given by

$$\begin{aligned} \dot{x}_1 &= f_1(x_1,\ldots,x_n,u) \\ &\ \vdots \\ \dot{x}_n &= f_n(x_1,\ldots,x_n,u) \end{aligned} \tag{8}$$

where $x_i \in \mathbb{R}^{N_i}$, $u \in \mathbb{R}^M$ and $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + M} \to \mathbb{R}^{N_i}$ is continuous and for all $r \in \mathbb{R}_+$ it is locally Lipschitz continuous in $x = (x_1^{\mathrm{T}},\ldots,x_n^{\mathrm{T}})^{\mathrm{T}}$ uniformly in u for $|u| \leq r$.

Here x_i is the state of the ith subsystem, and u is considered as an external control variable.

Without loss of generality we may assume to have the same input for all systems, because we may consider u as partitioned $u = (u_1, \ldots, u_n)$, such that each u_i is the input for subsystem i only. Then each f_i is of the form $f_i(\ldots, u) = \tilde{f}_i(\ldots, P_i(u)) = \tilde{f}_i(\ldots, u_i)$ with some projection P_i.

If we consider individual systems, we treat the state x_j, $j \neq i$ as an independent input for x_i. Thus, we call the ith subsystem of (8) ISS, if there exist functions β_i of class $\mathcal{KL}$ and $\gamma_{ij}, \gamma_i \in \mathcal{K} \cup \{0\}$ with $\gamma_{ii} = 0$, such that the solution $x_i(t)$ starting at $x_i(0)$ satisfies

$$|x_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{j=1}^{n} \gamma_{ij}(||x_{j\,[0,t]}||_\infty) + \gamma_i(||u||_\infty) \tag{9}$$

for all $t \geq 0$. We write $\Gamma^{\text{ISS}} = (\gamma_{ij})$. The functions γ_{ij} and γ_i are called (nonlinear) gains.

We also need GS and AG definitions for the subsystems:

Subsystem i of (8) is AG, if there exist $\gamma_{ij}, \gamma_i, \in \mathcal{K}_\infty \cup \{0\}$, $\gamma_{ii} = 0$, $j = 1, \ldots, n$, such that

$$\limsup_{t\to\infty} |x_i(t, \xi; x_j(\cdot), j \neq i, u(\cdot))| \leq \sum_j \gamma_{ij}(||x_j||_\infty) + \gamma_i(||u||_\infty), \tag{10}$$

and it is GS, if there exist $\gamma_{1i}, \sigma_{2ij}, \sigma_{3i} \in \mathcal{K}_\infty \cup \{0\}$, $\sigma_{2ii} = 0$, $i, j = 1, \ldots, n$, such that

$$|x_i(t, \xi; x_j(\cdot), j \neq i, u(\cdot))| \leq \sigma_{1i}(|\xi_i|) + \sum_j \sigma_{2ij}(||x_j||_\infty) + \gamma_{3i}(||u||_\infty), \tag{11}$$

where both (10) and (11) are supposed to hold for all $\xi_i \in \mathbb{R}^{N_i}$; $u(\cdot) \in L^\infty([0, \infty); \mathbb{R}^M)$ and $x_j(\cdot) \in L^\infty([0, \infty); \mathbb{R}^{N_j})$ for all $j \neq i$ and all $t \geq 0$.

We denote $\Gamma^{\text{AG}} = (\gamma_{ij})_{i,j=1}^n$ and $\Gamma^{\text{GS}} = (\sigma_{2ij})_{i,j=1}^n$. Let Γ be one of Γ^{ISS}, Γ^{AG}, or Γ^{GS}. This matrix is referred to as *gain matrix*.

Note that Γ defines in a natural way a map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$\Gamma\left((s_1, \ldots, s_n)^{\text{T}}\right) := \left(\sum_{j=1}^{n} \gamma_{1j}(s_j), \ldots, \sum_{j=1}^{n} \gamma_{nj}(s_j)\right)^{\text{T}} \tag{12}$$

for $s = (s_1, \ldots, s_n)^{\text{T}} \in \mathbb{R}^n_+$. Note that by the properties of γ_{ij} for $s_1, s_2 \in \mathbb{R}^n_+$ we have the implication

$$s_1 \geq s_2 \;\Rightarrow\; \Gamma(s_1) \geq \Gamma(s_2), \tag{13}$$

so that Γ defines a monotone operator on $\mathbb{R}^n_+$.

For vector valued functions $x = (x_1^T, \dots, x_n^T)^T : \mathbb{R}_+ \to \mathbb{R}^{N_1+\dots+N_n}$ such that $x_i : \mathbb{R}_+ \to \mathbb{R}^{N_i}$, $i = 1, \dots, n$, and times $0 \le t_1 \le t_2$, $t \ge 0$ we define

$$\left\| x_{[t_1,t_2]} \right\| := \begin{pmatrix} \|x_{1,[t_1,t_2]}\|_\infty \\ \vdots \\ \|x_{n,[t_1,t_2]}\|_\infty \end{pmatrix} \in \mathbb{R}_+^n \quad \text{and} \quad |x(t)| := \begin{pmatrix} |x_1(t)| \\ \vdots \\ |x_n(t)| \end{pmatrix} \in \mathbb{R}_+^n.$$

For $u(\cdot) \in L^\infty([0,\infty); \mathbb{R}^M)$, finite times $t \ge 0$, and for $s \in \mathbb{R}_+^n$ we introduce the abbreviating notation

$$\gamma(\|u\|_\infty) := \begin{pmatrix} \gamma_1(\|u\|_\infty) \\ \vdots \\ \gamma_n(\|u\|_\infty) \end{pmatrix} \in \mathbb{R}_+^n \quad \text{and} \quad \beta(s,t) := \begin{pmatrix} \beta_1(s_1,t) \\ \vdots \\ \beta_n(s_n,t) \end{pmatrix},$$

where $\gamma_1, \dots, \gamma_n$ come from (9) or (10) and β_i from (9).

Now we can rewrite the ISS conditions (9) of the subsystems in a vectorized form for $t \ge 0$ as

$$|x(t)| \le \beta(|x(0)|, t) + \Gamma^{\text{ISS}}(\| x_{[0,t]} \|) + \gamma(\|u\|_\infty). \tag{14}$$

The AG conditions now reads

$$\limsup_{t\to\infty} |x(t)| \le \Gamma^{\text{AG}}(\| x_{[0,\infty]} \|) + \gamma(\|u\|_\infty), \tag{15}$$

where this time γ denotes the vector $[\gamma_1, \dots, \gamma_n]^T$ corresponding to (10).

For the GS assumption we obtain

$$|x(t)| \le \sigma_1(|\xi|) + \Gamma^{\text{GS}}(\| x_{[0,t]} \|) + \gamma_3(\|u\|_\infty), \tag{16}$$

where $\sigma_1(|\xi|) = (\sigma_{11}(|\xi_1|), \dots, \sigma_{1n}(|\xi_n|))^T$ and $\gamma_3(\|u\|_\infty) = (\gamma_{31}(\|u\|_\infty), \dots, \gamma_{3n}(\|u\|_\infty))^T$.

Assuming each of the subsystems of (8) to be ISS, we are interested in conditions guaranteeing that the whole system defined by $x = \left(x_1^T, \dots, x_n^T\right)^T$, $f = \left(f_1^T, \dots, f_n^T\right)^T$ and

$$\dot{x} = f(x,u) \tag{17}$$

is ISS (from u to x).

Remark 6 (Additional Preliminaries) We also need some notation from lattice theory, cf. [28] for example.

Let $(\mathbb{R}_+^n, \sup, \inf)$ be the lattice given by the standard partial order defined by (2). Here sup denotes the supremum and inf denotes the infimum with respect to the partial order. With this notation the upper limit for bounded functions $s : \mathbb{R}_+ \to \mathbb{R}_+^n$ is defined by

$$\limsup_{t\to\infty} s(t) := \inf_{t\geq 0} \sup_{\tau\geq t} s(\tau).$$

We will need the following property.

Lemma 7 *Let* $s : \mathbb{R}_+ \to \mathbb{R}^n_+$ *be continuous and bounded. Then*

$$\limsup_{t\to\infty} s(t) = \limsup_{t\to\infty} \left\| s_{[t/2,\infty)} \right\|.$$

Proof Let $\limsup_{t\to\infty} s(t) =: a \in \mathbb{R}^n_+$ and $\limsup_{t\to\infty} \left\| s_{[t/2,\infty)} \right\| =: b \in \mathbb{R}^n_+$. For every $\varepsilon \in \mathbb{R}^n_+$, $\varepsilon > 0$ (component-wise!) there exist $t_a, t_b \geq 0$ such that

$$\forall\, t \geq t_a : \sup_{t\geq t_a} s(t) \leq a + \varepsilon \quad \text{and} \quad \forall\, t \geq t_b : \sup_{t\geq t_b} \left\| s_{[t/2,\infty)} \right\| \leq b + \varepsilon. \tag{18}$$

Clearly we have

$$s(t) \leq \left\| s_{[t/2,\infty)} \right\|$$

for all $t \geq 0$, i.e., $a \leq b$. On the other hand $s(\tau) \leq a + \varepsilon$ for $\tau \geq t$ implies $\left\| s_{[\tau/2,\infty)} \right\| \leq a + \varepsilon$ for $\tau \geq 2t$, i.e., $b \leq a$.

4 Main results

In the following subsection we present a nonlinear version of the small gain theorem for networks. In Subsect. 4.5 we restate this theorem for the case when the gains are linear functions.

4.1 AG, GS, and ISS small gain theorems

We introduce the following notation. For $\alpha_i \in \mathcal{K}_\infty, i = 1, \ldots, n$ define a diagonal operator $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$D\left((s_1, \ldots, s_n)^{\mathrm{T}}\right) := \begin{pmatrix} (\mathrm{Id} + \alpha_1)(s_1) \\ \vdots \\ (\mathrm{Id} + \alpha_n)(s_n) \end{pmatrix}. \tag{19}$$

We say that a gain matrix Γ satisfies the *small gain condition*, if there exists an operator D as in (19), such that for all $s \in \mathbb{R}^n_+$, $s \neq 0$ we have

$$\Gamma \circ D(s) := \Gamma(D(s)) \not\geq s. \tag{20}$$

For short we just write $\Gamma \circ D \not\geq \mathrm{id}$. This condition can equivalently be stated as: there has to be at least one $i \in \{1, \dots, n\}$, such that the ith component of $\Gamma \circ D$ decreases, i.e., $\Gamma(D(s))_i < s_i$. In other words (20) states that there exists no $s \geq 0,\ s \neq 0$, such that $\Gamma \circ D(s) \geq s$.

Note that $\Gamma \circ D \not\geq \mathrm{id}$ if and only if $D \circ \Gamma \not\geq \mathrm{id}$.

Theorem 8 (small gain global stability theorem) *Assume each subsystem of* (8) *is GS. If* Γ^{GS} *satisfies the small gain condition* (20)*, then* (17) *possesses the global stability property.*

Theorem 9 (small gain theorem for asymptotic gains) *Assume each subsystem of* (8) *is AG and solutions of the composite system* (17) *exist for all times and are essentially bounded. If* Γ^{AG} *satisfies the small gain condition* (20)*, then* (17) *has the asymptotic gain property.*

Note that here we explicitly have to ask for the existence of trajectories for all times. This is in general not guaranteed by the AG condition, as Example 14 in Sect. 4.2 shows.

Before proving Theorems 8 and 9 we note some immediate consequences. Combining the two theorems we obtain:

Theorem 10 *Assume each subsystem of* (8) *is GS and AG. If both* Γ^{AG} *and* Γ^{GS} *satisfy the small gain condition* (20)*, then* (17) *has the global stability property and the asymptotic gain property.*

Since AG and GS together are equivalent to ISS, this can be reformulated to obtain our main result:

Theorem 11 (small gain theorem for networks) *Consider the system* (8) *and suppose that each subsystem is ISS, i.e., condition* (9) *holds for all* $i = 1, \dots, n$. *Let* Γ *be given by* (12)*. If there exists a mapping* D *as in* (19)*, such that*

$$(\Gamma \circ D)(s) \not\geq s, \quad \forall s \in \mathbb{R}^n_+ \setminus \{0\},$$

then the system (17) *is ISS from* u *to* x.

Proof (of Theorem 11) The ISS condition $|x(t)| \leq \beta(|\xi|, t) + \gamma_{\mathrm{ISS}}(\|u\|_\infty)$ implies $|x(t)| \leq \beta(|\xi|, 0) + \gamma_{\mathrm{ISS}}(\|u\|_\infty)$, where $\beta(\cdot, 0)$ is a class $\mathcal{K}_\infty$-function. But this is just a GS estimate, implying that $\sigma_2 \leq \gamma_{\mathrm{ISS}}$ for GS estimates σ_2. This gives $\Gamma \geq \Gamma^{\mathrm{GS}}$ and makes Theorem 8 applicable. In particular this shows that solutions exist for all times and are uniformly bounded.

Obviously, by looking at (14) and (15), ISS implies AG, since given an ISS gain γ_{ISS}, it is also a valid asymptotic gain. So we may assume $\Gamma \geq \Gamma^{\mathrm{AG}}$ and Theorem 9 is applicable.

Together we obtain that the interconnection is GS and AG, hence it is ISS by a result in [27]. □

Remark 12 Condition (20) is equivalent to the requirement that the graph $\{(s, \Gamma \circ D(s)) : s \in \mathbb{R}^n_+\}$ of the map $\Gamma \circ D$ does not intersect the set $\{(s, w) \in \mathbb{R}^n_+ \times \mathbb{R}^n_+ : w \geq s > 0\}$. Although looking very complicated to handle at first sight, this small gain condition is a straightforward extension of the ISS small gain theorem of [13] for two systems (and coincides with it in this case), see Sect. 5.1 for details. This condition has several interesting interpretations, as we will discuss in Sect. 5.

The following lemma provides the essential argument for the proofs of the above theorems:

Lemma 13 *Let $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ satisfy the small gain condition* (20). *Then there exists a $\varphi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,*

$$(Id - \Gamma)(w) \leq v \tag{21}$$

implies $|w| \leq \varphi(|v|)$.

Proof Let D be given by the small gain condition. Fix $v \in \mathbb{R}^n_+$. We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (21) at least some components have to be bounded. To this end let

$$\begin{aligned} r^* &:= (D - \mathrm{Id})^{-1}(v) = \begin{pmatrix} \alpha_1^{-1}(v_1) \\ \vdots \\ \alpha_n^{-1}(v_n) \end{pmatrix} \\ \text{and } s^* &:= D(r^*) = \begin{pmatrix} v_1 + \alpha_1^{-1}(v_1) \\ \vdots \\ v_n + \alpha_n^{-1}(v_n) \end{pmatrix}. \end{aligned} \tag{22}$$

We claim that $s \geq s^*$ implies that $w = s$ does not satisfy (21). So let $s \geq s^*$ be arbitrary and $r = D^{-1}(s) \geq r^*$ (as $D^{-1} \in \mathcal{K}^n_\infty$). For such s we have

$$s - D^{-1}(s) = D(r) - r \geq D(r^*) - r^* = v,$$

where we have used that $(D - \mathrm{Id}) \in \mathcal{K}^n_\infty$. The assumption that $w = s$ satisfies (21) leads to

$$s \leq v + \Gamma(s) \leq s - D^{-1}(s) + \Gamma(s),$$

or equivalently, $0 \leq \Gamma(s) - D^{-1}(s)$. This implies for $r = D^{-1}(s)$ that

$$r \leq \Gamma \circ D(r),$$

in contradiction to (20). This shows that the set of $w \in \mathbb{R}^n_+$ satisfying (21) does not intersect the set

$$Z_1 := \{w \in \mathbb{R}^n_+ \mid w \geq s^*\}.$$

Assume now that $w \in \mathbb{R}^n_+$ satisfies (21). Let $s^1 := s^*$. If $s^1 \not\geq w$, then there exists an index set $I_1 \subset \{1, \ldots, n\}$, possibly depending on w, such that

$$w_i > s^1_i, \text{ for } i \in I_1 \quad \text{and} \quad w_i \leq s^1_i, \text{ for } i \in I_1^c := \{1, \ldots, n\} \setminus I_1.$$

So from (21) we obtain

$$\begin{bmatrix} w_{I_1} \\ w_{I_1^c} \end{bmatrix} - \begin{bmatrix} \Gamma_{I_1 I_1} & \Gamma_{I_1 I_1^c} \\ \Gamma_{I_1^c I_1} & \Gamma_{I_1^c I_1^c} \end{bmatrix} \left(\begin{bmatrix} w_{I_1} \\ w_{I_1^c} \end{bmatrix} \right) \leq \begin{bmatrix} v_{I_1} \\ v_{I_1^c} \end{bmatrix}.$$

Hence we have in particular

$$\begin{aligned} w_{I_1} - \Gamma_{I_1 I_1}(w_{I_1}) &\leq v_{I_1} + \Gamma_{I_1 I_1^c}(s^1_{I_1^c}) \\ &\leq \underbrace{D_{I_1} \circ (D_{I_1} - \mathrm{Id}_{I_1})^{-1}}_{>\mathrm{Id}} \circ (v_{I_1} + \Gamma_{I_1 I_1^c}(s^1_{I_1^c})) =: s^2_{I_1}. \end{aligned} \tag{23}$$

Note that $\Gamma_{I_1 I_1}$ satisfies (20) with D replaced by D_{I_1}. Thus, arguing just as before, we obtain, that $w_{I_1} \geq s^2_{I_1}$ is not possible. Hence some more components of w must be bounded.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s^{j+1}_i\},$$

with $I^c_{j+1} := \{1, \ldots, n\} \setminus I_{j+1}$ and

$$s^{j+1}_{I_j} := D_{I_j} \circ (D_{I_j} - \mathrm{Id}_{I_j})^{-1} \circ (v_{I_j} + \Gamma_{I_j I_j^c}(s^j_{I_j^c})).$$

This nesting will end after at most $n-1$ steps: There exists a maximal $k \leq n$, such that

$$\{1, \ldots, n\} \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_k \neq \emptyset$$

and all components of w_{I_k} are bounded by the corresponding components of $s^{k+1}_{I_k}$. Let

$$s_\zeta := \max\{s^*, R_{I_1}(s^1), \ldots, R_{I_k}(s^k)\}.$$

If we denote by $[M]^n$ the n-fold composition $M \circ \cdots \circ M$, then for w satisfying (21) we clearly have

$$w \leq s_\zeta \leq [D \circ (D - \mathrm{Id})^{-1} \circ (\mathrm{Id} + \Gamma)]^n(v)$$

and the term completely on the right-hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (21) also satisfies

$$w \leq [D \circ (D - \mathrm{Id})^{-1} \circ (\mathrm{Id} + \Gamma)]^n \left(|v|_{\max}, \ldots, |v|_{\max}\right)^{\mathrm{T}}$$

and taking the maximum-norm on both sides yields

$$|w|_{\max} \leq \varphi(|v|_{\max})$$

for some function φ of class $\mathcal{K}_\infty$. This completes the proof of the lemma.

We proceed with the proof of Theorems 8 and 9.

Proof (of Theorem 8) We may assume that the vectorized formulation (16) of the GS conditions (11) for all $\xi \in \mathbb{R}^{N_1+\cdots+N_n}$, and $u(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{R}^M)$ is satisfied for some $t > 0$:

$$\llbracket x(t) \rrbracket \leq \sigma_1(\llbracket \xi \rrbracket) + \Gamma^{\text{GS}}(\left\| x_{[0,t]} \right\|) + \gamma_3(\|u\|_\infty).$$

Taking the supremum on both sides over $\tau \in [0, t]$ we obtain

$$(\text{Id} - \Gamma^{\text{GS}})\left(\left\| x_{[0,t]} \right\|\right) \leq \quad \sigma_1(\llbracket \xi \rrbracket) + \gamma_3(\|u\|_\infty).$$

Note, that the right-hand side is independent of $t > 0$, hence this estimate holds for all $t > 0$. Now by Lemma 13 we find

$$\begin{aligned} \|x_{[0,t]}\|_\infty &\leq \varphi\left(|\sigma_1(\llbracket \xi \rrbracket) + \gamma_3(\|u\|_\infty)|\right) \\ &\leq \varphi\left(2 \cdot |\sigma_1(\llbracket \xi \rrbracket)|\right) + \varphi\left(2 \cdot |\gamma_3(\|u\|_\infty)|\right) \qquad (24) \\ &=: s_\infty \qquad (25) \end{aligned}$$

for some class $\mathcal{K}$ function φ and all times $t \geq 0$. Hence for every initial condition and essentially bounded input u the solution of our system (17) exists for all times $t \geq 0$ and is uniformly bounded, since s_∞ in (25) does not depend on t. The GS estimate for (17) is then given by (24).

Proof (of Theorem 9) Assume solutions of (17) with essentially bounded input exist for all times, are uniformly bounded, and the AG formulation (15) is valid, i.e., we have for all $\tau \geq 0$, all initial conditions, and all inputs $u(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{R}^M)$ that

$$\limsup_{t\to\infty} \llbracket x(t) \rrbracket \leq \Gamma^{\text{AG}}(\left\| x_{[\tau,\infty)} \right\|) + \gamma(\|u\|_\infty). \qquad (26)$$

By Lemma 7 we have

$$\limsup_{t\to\infty} \llbracket x(t) \rrbracket = \limsup_{\tau\to\infty} \left\| x_{[\tau,\infty)} \right\| =: l(x) \in \mathbb{R}^n_+,$$

hence by (5) inequality (26) can be stated equivalently as

$$l(x) \leq \Gamma^{\text{AG}}(l(x)) + \gamma(\|u\|_\infty),$$

where we used (13). It follows that

$$(\text{Id} - \Gamma^{\text{AG}}) \circ l(x) \leq \gamma(\|u\|_\infty).$$

Finally, by Lemma 13 we have

$$|l(x)| \leq \varphi(|\gamma(\|u\|_\infty)|) \tag{27}$$

for some φ of class $\mathcal{K}_\infty$, which is the desired asymptotic gain property (4).

4.2 AG and forward completeness of interconnections

In Theorem 9 we explicitly require solutions of the interconnection to exist for all times. If this requirement is not met, then the assertion of the theorem is false, as the following example shows. It is based on an example in [10].

Example 14 Consider the planar autonomous system defined by

$$\dot{x} = f(x, y) := \frac{x^2(y - x) + y^5}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} \tag{28}$$

$$\dot{y} = g(x, y) := \frac{y^2(y - 2x)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)}. \tag{29}$$

It is shown in [10, Sect. 40, pp. 191–194] that for (28) and (29) the origin is globally attractive but unstable.

Now replace (29) by

$$\dot{y} = \tilde{g}(x, y, v) := \frac{y^2(y - 2x)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} \exp(v^2), \tag{30}$$

to obtain the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ \tilde{g}(x, y, v) \end{pmatrix}, \tag{31}$$

where $v \in \mathbb{R}$ is a parameter.

By the same methods as in [10, Sect. 40, pp. 191–194] it can be shown that for any fixed $v \in \mathbb{R}$ the origin for system (31) is unstable, but globally attractive. By symmetry it suffices to consider the open upper half plane in $\mathbb{R}^2$. The proof is exactly as in [10] and therefore omitted.

Next we show, that for $u(\cdot) \in L^\infty(\mathbb{R}_+; \mathbb{R})$, $\|u\|_\infty < m$, the origin of the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} f(x, y) \\ \tilde{g}(x, y, u) \end{pmatrix} \tag{32}$$

is globally attractive and unstable: Suppose a trajectory $\varphi(\cdot, (x_0, y_0)^T, u(\cdot))$ of (32) starts in $(x_0, y_0)^T$ lying in

$$\Psi^+ := \{(x, y)^T \in \mathbb{R}^2 : (x \leq 0 \text{ and } y > 0) \text{ or } (x > 0 \text{ and } y > 2x)\}.$$

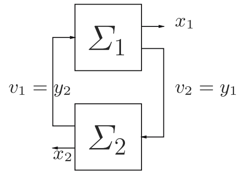

Fig. 1 Feedback system (33)

In the region Ψ^+ the second component of the trajectory $\varphi(\cdot) := \varphi(\cdot, (x_0, y_0)^T, u(\cdot))$ is bounded by the second component of the trajectory $\eta(\cdot) := \eta(\cdot, (x_0, y_0)^T, ||u||_\infty)$ of system (31) in the sense that for all $t, \tau \geq 0$ such that $\varphi(t), \eta(\tau) \in \Psi^+$ we have

$$\varphi(t)_1 = \eta(\tau)_1 \quad \Rightarrow \quad \varphi(t)_2 \leq \eta(\tau)_2.$$

This follows as for almost all t we have $\tilde{g}(x, y, u(t)) \leq \tilde{g}(x, y, \|u\|_\infty)$.

Now $\eta(\cdot)$ is known to reach the half-line $H := \{(x, y)^T \in \mathbb{R}^2 : x \geq 0, y = 2x\}$ in finite time (using the argument provided in [10]). Since in Ψ^+ we have $\dot{x} > 0$ and $\dot{y} > 0$, the trajectory $\varphi(\cdot, (x_0, y_0)^T, u)$ also hits the line H in finite time $\tau > 0$ at the point $(x_\tau, y_\tau)^T$, see Fig. 2.

Define the "attractive set" $A := \{(x, y)^T \in \mathbb{R}^2 : y \geq 0, x > \frac{1}{2}y\}$. Trajectories with bounded input starting on the line $H \subset \overline{A}$ are ultimately attracted to the origin: on H every velocity vector points into A, in A the y-component of each velocity vector is strictly negative. At some point each trajectory in A reaches a neighborhood of the x-axis, where all velocity vectors point to the lower left.

The preceding argument shows, that system (32) has the asymptotic gain property, with arbitrary small gain γ. So with slight abuse of notation we may choose $\gamma = 0$ as the asymptotic gain.

Now consider the feedback interconnection of two systems (32), each with input $v = y$ as depicted in Fig. 1:

$$\Sigma_1 : \begin{pmatrix} \dot{x}_1 \\ \dot{y}_1 \end{pmatrix} = \begin{pmatrix} f(x_1, y_1) \\ \tilde{g}(x_1, y_1, y_2) \end{pmatrix} \qquad \Sigma_2 : \begin{pmatrix} \dot{x}_2 \\ \dot{y}_2 \end{pmatrix} = \begin{pmatrix} f(x_2, y_2) \\ \tilde{g}(x_2, y_2, y_1) \end{pmatrix} \tag{33}$$

We concentrate only on the "diagonal", that is, we let each system start at $(x_0, y_0)^T \in \mathbb{R}^2$ and denote $x_1 = x_2 = x$ and $y_1 = y_2 = y$. Hence the evolution of both systems is exactly the same, motivating the term diagonal. The small gain condition, which in this case reduces to $\gamma^2 \equiv 0 < \mathrm{id}$, is clearly satisfied.

On the diagonal the dynamics are given by

$$\dot{x} = f(x, y), \quad \dot{y} = \tilde{g}(x, y, y). \tag{34}$$

For suitable initial conditions system (34) has a finite escape time, as the next calculation shows: let $V(x, y) = \frac{1}{2}y^2$ be a storage function on the set $\{(x, y)^T \in \mathbb{R}^2 : y > 0\} \cup \{(0, 0)^T\}$. We will show that $\dot{V} > V^2$ along trajectories for suitable initial conditions. This implies that not all trajectories of the feedback system exist for all times.

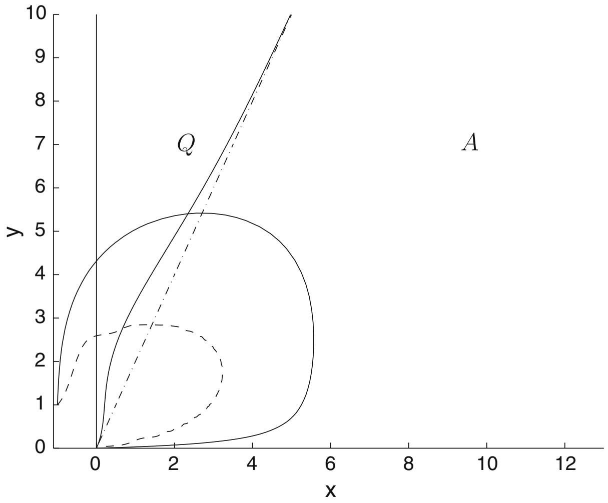

Fig. 2 A trajectory of the (28)–(30) in Example 2: the *dashed* trajectory starts in $(-1, 1)^{\mathrm{T}}$ with input $\sqrt{3}\sin(t)$. The *solid* trajectory starts in the same point with constant input equal to $\sqrt{3}$. The half-line H with (slope 2) is dashed dotted. The regions Q and A are depicted as well

First we define the set $Q := \left\{(x, y)^{\mathrm{T}} \in \mathbb{R}^2 : y > 0, 0 \le x < \frac{y(e^y - y^2 - 1)}{2(e^y - 1)}\right\}$, see Fig. 2. On Q we have

$$\dot{x} < \frac{1}{2}\dot{y} \tag{35}$$

for system (33). Asymptotically we have $\frac{y(e^y - y^2 - 1)}{2(e^y - 1)} \approx \frac{1}{2}y$ as $y \to \infty$. Also $\frac{y(e^y - y^2 - 1)}{2(e^y - 1)} < \frac{1}{2}y$ for all $y > 0$.

Now we show that in the feedback system (33) a trajectory starting in $(x, y) \in Q$, with $y > 6$, will always stay in Q (as long as it exists): To this end consider the horizontal distance between $\varphi(\cdot, (x_0, y_0)^{\mathrm{T}})$ and the point $\begin{pmatrix} \frac{1}{2}\varphi(t, (x_0, y_0)^{\mathrm{T}})_2 \\ \varphi(t, (x_0, y_0)^{\mathrm{T}})_2 \end{pmatrix}$ on the half line H. This distance is just

$$d(t) = \frac{1}{2}\varphi(t, (x_0, y_0)^{\mathrm{T}})_2 - \varphi(t, (x_0, y_0)^{\mathrm{T}})_1.$$

By (35) the derivative of $d(t)$ satisfies

$$\dot{d}(t) = \frac{1}{2}\tilde{g}(x, y, y) - f(x, y) > 0, \tag{36}$$

for $(x, y)^{\mathrm{T}} \in Q$. Hence $d(t)$ is non decreasing on the set Q. By (28) and (30) we have $\dot{x} > 0$ and $\dot{y} > 0$ on Q. Moreover, the right hand side boundary of Q approaches H, i.e., the distance between this boundary and H decays monotonically for increasing values of $y > 6$. Hence the solution $\varphi(\cdot, (x_0, y_0)^{\mathrm{T}})$ for $(x_0, y_0)^{\mathrm{T}} \in Q$ must stay in Q, as long as it exists.

Now assume $(x, y)^{\mathrm{T}} \in Q$ with $y \gg 1$ sufficiently large and such that $y > 2x + 1$. By (36) if follows that $\varphi_2(t) > 2\varphi_1(t) + 1$ for all $t \geq 0$ along the trajectory φ with initial condition (x, y). Because of the dominating exponential term we have

$$\begin{aligned}\nabla V(x, y)(f(x, y), \tilde{g}(x, y, y))^{\mathrm{T}} &= \frac{y^3(y - 2x)\exp(y^2)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} \\ &> \frac{y^3 \exp(y^2)}{(x^2 + y^2)(1 + (x^2 + y^2)^2)} > \frac{1}{4}y^4 = (V(x, y))^2\end{aligned}$$

along trajectories, provided that y is sufficiently large. Finally, $\dot{V} > V^2$ implies a finite escape time for the y-component.

Concluding, this example shows that although systems have the asymptotic gain property and the small gain condition is satisfied, Theorem 9 may not be applicable to the feedback interconnection, since solutions of the interconnected system need not exist for all times.

4.3 The maximum formulation of ISS

The ISS estimates for (8) can be formulated equivalently using the maximum instead of a summation. Using the maximum the condition for ISS is

$$\|x_i(t, \xi_i^0, x_j, j \neq i, u)\| \leq \max_j \{\beta(\|\xi_i^0\|, t), \gamma_{ij}(\|x_j\|_\infty), \gamma_u(\|u\|_\infty)\}, \tag{37}$$

for $i = 1, \ldots, n$, all initial conditions and all measurable bounded inputs. Note that to obtain such an estimate for a particular system the gains γ_{ij}, γ_u are in general different from the gains that would be used in an ISS inequality using summation as in (9). Still we can write down the gain matrix $\Gamma = \Gamma^{\mathrm{ISS}}$ as we did before and define the max-operator $\Gamma_\otimes$ by

$$\Gamma_\otimes(s)_i := \max_j \{\gamma_{ij}(s_j)\}. \tag{38}$$

If the gains γ_{ij} are all linear, then $\Gamma_\otimes$ is a max linear operator [19]. Note that this is not the same as a max-plus linear operator.

In the discrete time context Teel [31] proves that if we have maximum ISS estimates of the type (37) for each subsystem and if for each cycle (or equivalently, each minimal cycle) in Γ we have

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \cdots \circ \gamma_{k_{p-1}k_p} < \mathrm{id},$$

for all $(k_1, \ldots, k_p) \in \{1, \ldots, n\}^p$, $p \geq 1$ where $k_1 = k_p$, then the network under consideration is input-to-state stable. This result extends in a straightforward manner to continuous time systems. It is an easy exercise to show that the cycle condition and the statement

$$\Gamma_\otimes(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n, s \neq 0,$$

are equivalent. Note that this relies crucially on the operation defined in (38) and does not hold for the operator Γ defined earlier.

Potrykus et al. [21] prove a similar but rather involved small gain theorem for continuous time systems; their small gain condition is not at first sight equivalent to the one in [31].

For real non negative matrices it is known that the linear cycle condition and that the max spectral radius is less than unity are equivalent, a nice survey on this topic can be found in Lur [19]. In general $\mu(A) \leq \rho(A)$, where $\mu(A)$ is the maximal cycle geometric mean of a non negative matrix A (corresponding to the cycle condition), and $\rho(A)$ is the usual spectral radius of A, c.f. [19].

Now given a gain matrix $\Gamma = (\gamma_{ij})_{i,j=1}^n$ we associate two operators $\mathbb{R}_+^n \to \mathbb{R}_+^n$, namely the one given by $\Gamma(s)_i = \sum_j \gamma_{ij}(s_j)$ and the one given by $\Gamma_\otimes(s)_i = \max_j \gamma_{ij}(s_j)$, where $s \in \mathbb{R}_+^n$, $i = 1, \ldots, n$. The small gain condition

$$\Gamma \not\geq \mathrm{id} \tag{39}$$

is a stronger condition on the gains than

$$\Gamma_\otimes \not\geq \mathrm{id}, \tag{40}$$

both are sufficient conditions for the max-formulation of the ISS small gain theorem for networks, but the latter is not sufficient for the sum-formulation. For the case $n = 2$ with $\gamma_{11} = \gamma_{22} = 0$, both conditions are equivalent, but for $n \geq 3$ condition (39) is strictly stronger than condition (40), as can be shown easily by a linear example.

4.4 Discrete time systems

We now briefly discuss the extension of our result to the discrete time setting. In [12] Jiang et al. prove an ISS small gain theorem for discrete time systems, while in [15] they derive a small gain theorem for locally input-to-state stable systems. Both papers use the maximum formulation of ISS.

In [17] Laila and Nešić consider parameterized discrete time systems that are semi globally practically ISS and give a small gain theorem for two such systems in feedback interconnection.

Input-to-state stability for discrete time systems is defined in analogy to the continuous time case, but with time $\mathbb{R}_+$ replaced by $\mathbb{N} = \{0, 1, 2, \ldots\}$:

In this subsection we denote by $[0, k]$ the set $\{0, \ldots, k\}$ and by $\|x\|_\infty = \sup_{l\in\mathbb{N}}\{x_l\}$ for functions $x : \mathbb{N} \to \mathbb{R}^N$.

Consider the interconnected discrete time system

$$\Sigma : \begin{cases} \Sigma_1 : x_1(k+1) = f_1(x_1(k), \ldots, x_n(k), u(k)) \\ \qquad\qquad \vdots \\ \Sigma_n : x_n(k+1) = f_n(x_1(k), \ldots, x_n(k), u(k)) \end{cases} \quad \text{for } k \in \mathbb{N}, \tag{41}$$

where $x_i(k) \in \mathbb{R}^{N_i}$, $u(k) \in \mathbb{R}^M$, and $f_i : \mathbb{R}^{\sum_{j=1}^n N_j + M} \to \mathbb{R}^{N_i}$ is continuous.

System Σ_i is ISS, if there exists $\beta \in \mathcal{KL}$ and $\gamma_{ij}, \gamma \in \mathcal{K} \cup \{0\}$ with $\gamma_{ii} = 0$, such that every solution $x_i : \mathbb{N} \to \mathbb{R}^{N_i}$ of (41) satisfies

$$|x_i(k)| \leq \beta_i(|x_i(0)|, k) + \sum_{j=1}^{n} \gamma_{ij}(||x_{j\,[0,k]}||_\infty) + \gamma(||u||_\infty) \tag{42}$$

for all inputs $x_j : \mathbb{N} \to \mathbb{R}^{N_j}$, $j = 1, \ldots, n$, $j \neq i$, and $u : \mathbb{N} \to \mathbb{R}^M$.

As a corollary of Theorem 11 we easily obtain the next result:

Proposition 15 *Consider the systems Σ_i in (41) and suppose that each subsystem is ISS, i.e., condition (42) holds for all $i = 1, \ldots, n$. Let $\Gamma = (\gamma_{ij})_{i,j=1}^n$. If there exists a mapping D as in (19), such that*

$$(\Gamma \circ D)(s) \not\geq s, \quad \forall s \in \mathbb{R}_+^n \setminus \{0\},$$

then the system Σ in (41) is ISS from u to x.

Proof The assertion of Theorem 11 still is true if we replace system (8) by the discrete time system (41) and the definition of ISS by (42). Note that the proofs of Theorems 8, 9, and 11 do not depend on the time set being a continuum or discrete set.

4.5 Linear gains

Suppose the gain functions γ_{ij} are all linear, hence Γ is a linear mapping and (12) is just matrix-vector multiplication. In this case the network small gain condition (20) has a straightforward interpretation in terms of well known properties of matrices. The spectral radius of a matrix M is denoted by $\rho(M)$. For non-negative matrices Γ it is well known (see, e.g., [3, Theorem 2.1.1, p. 26, and Theorem 2.1.11, p. 28]) that

(i) $\rho(\Gamma)$ is an eigenvalue of Γ with a corresponding non-negative eigenvector,
(ii) if $\alpha x \leq \Gamma x$ holds for some $x \in \mathbb{R}_+^n \setminus \{0\}$ then $\alpha \leq \rho(\Gamma)$.

Using this it is easy to see that for a non-negative matrix $\Gamma \in \mathbb{R}^{n \times n}$ the following are equivalent:

(i) $\rho(\Gamma) < 1$,
(ii) $\forall s \in \mathbb{R}_+^n \setminus \{0\} : \Gamma s \not\geq s$,
(iii) $\Gamma^k \to 0$, for $k \to \infty$,
(iv) there exist $a_1, \ldots, a_n > 0$ such that $\forall s \in \mathbb{R}_+^n \setminus \{0\}$:

$$\Gamma(I + \mathrm{diag}(a_1, \ldots, a_n))s \not\geq s.$$

Note that (iv) is the linear version of (20). As condition (i) makes no sense in the nonlinear setting, (20) has been used as an extension to the nonlinear case. As a consequence we obtain

Corollary 16 *Consider n interconnected ISS systems as in the previous section on the problem description with a linear gain matrix Γ, such that for the spectral radius ρ of Γ we have*

$$\rho(\Gamma) < 1. \tag{43}$$

Then the system defined by (17) *is ISS from u to x.*

Remark 17 For the case of large-scale interconnected input-output systems a result similar to Corollary 16 exists, cf. [32, p. 110]. It also covers Corollary 16 as a special case. The condition on the spectral radius is quite the same and is applied to a matrix, whose entries are finite gains of products of *interconnection operators* and corresponding *subsystem operators*. These gains are non-negative numbers and, roughly speaking, defined as the minimal possible slope of affine bounds on the interconnection operators.

5 Interpretation of the generalized small gain condition

In this section we wish to provide insight into the small gain condition of Theorem 11. We are considering $\Gamma \in (\mathcal{K} \cup \{0\})^{n\times n}$ with zero diagonal as introduced in (12). The small gain condition sufficient for ISS of an interconnection defined with the maximum formulation is

$$\Gamma \ngeq \mathrm{id}, \tag{44}$$

whereas for the robust condition, we need that there exists a diagonal D as in (19) such that

$$\Gamma \circ D \ngeq \mathrm{id}, \tag{45}$$

in case of ISS defined via summation.

We first show, that Theorem 11 covers the known interconnection results for cascades and feedback interconnections and discuss implications for the case of linear systems.

Further, we investigate topological consequences of the small gain condition. Note that this condition has an interesting interpretation for the stability analysis of a discrete time dynamical system defined through the gain matrix Γ. Finally, an overview of all the interrelations is presented.

5.1 Connections to known results

As an easy consequence of Theorem 11 we recover, that an arbitrary feed forward cascade of ISS subsystems is again ISS. If the subsystems are enumerated consecutively and the gain function from subsystem j to subsystem $i > j$ is denoted by γ_{ij}, then the resulting gain matrix has non-zero entries only below the diagonal. For arbitrary $\alpha \in \mathcal{K}_\infty$ the gain matrix with entries $\gamma_{ij} \circ (\mathrm{Id}_{\mathbb{R}_+} + \alpha)$ for $i > j$ and 0 for $i \leq j$ clearly satisfies (20). Therefore any feed forward cascade of ISS systems is ISS.

Consider $n = 2$ in Eq. (8), i.e., two subsystems with linear gains. Then in Corollary 16 we have

$$\Gamma = \begin{bmatrix} 0 & \gamma_{12} \\ \gamma_{21} & 0 \end{bmatrix}, \quad \gamma_{ij} \in \mathbb{R}_+,$$

and $\rho(\Gamma) < 1$ if and only if $\gamma_{12}\gamma_{21} < 1$. Hence we obtain the known small gain theorem, cf. [9,14].

For nonlinear gains and $n = 2$ the condition (20) in Theorem 11 reads as follows: There exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\Gamma \circ D(s) = \begin{pmatrix} \gamma_{12} \circ (\mathrm{Id} + \alpha_2)(s_2) \\ \gamma_{21} \circ (\mathrm{Id} + \alpha_1)(s_1) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \end{pmatrix},$$

for all $s = (s_1, s_2)^{\mathrm{T}} \in \mathbb{R}^2_+$. This is easily seen to be equivalent to

$$\gamma_{12} \circ (\mathrm{Id} + \alpha_2) \circ \gamma_{21} \circ (\mathrm{Id} + \alpha_1)(t) < t, \quad \forall t > 0. \tag{46}$$

To this end we consider the vector $[\gamma_{12} \circ (\mathrm{Id} + \alpha_2)(t), t]^{\mathrm{T}}$ for any $t > 0$. Applying $\Gamma \circ D$ we see that the first component does not change, hence the second one must be less than t, which is just (46). The converse implication is obvious. Now inequality (46) is equivalent to the condition in the small gain theorem of [13], namely, that for some $\tilde{\alpha}_1, \tilde{\alpha}_2 \in \mathcal{K}_\infty$ it should hold that

$$(\mathrm{Id} + \tilde{\alpha}_1) \circ \gamma_{21} \circ (\mathrm{Id} + \tilde{\alpha}_2) \circ \gamma_{12}(s) \leq s, \quad \forall s > 0, \tag{47}$$

for all $s \in \mathbb{R}_+$, hence our theorem contains this result as a particular case.

Example 18 The condition (47) of [13] seems to be very similar to the small gain condition $\gamma_{12} \circ \gamma_{21}(s) < s$ of [9,14], however in these papers the maximum formulation of ISS is used, so that the interpretation of the gains is different. This similarity raises the question, whether the compositions with $(\mathrm{Id} + \tilde{\alpha}_i)$, $i = 1, 2$ in (47) or more generally with D in (20) are necessary in the context of the summation formulation of ISS. The following example shows that this is indeed the case.

Consider the equation

$$\dot{x} = -x + u(1 - e^{-u}), \quad x(0) = x^0 \in \mathbb{R}, \ u \in \mathbb{R}.$$

Integrating for $t \geq 0$ it follows

$$\begin{aligned} x(t) &= e^{-t}x^0 + \int_0^t e^{-(t-\tau)}u(\tau)(1 - e^{-u(\tau)})\,\mathrm{d}\tau \\ &\leq e^{-t}x^0 + ||u||_\infty(1 - e^{-||u||_\infty}) = e^{-t}x^0 + \gamma(\|u\|_\infty), \end{aligned}$$

with $\gamma(s) = s(1 - e^{-s})$. Clearly $\gamma(s) < s$. Then for a feedback system

$$\dot{x}_1 = -x_1 + x_2(1 - e^{-x_2}) + u(t), \tag{48}$$

$$\dot{x}_2 = -x_2 + x_1(1 - e^{-x_1}) + u(t) \tag{49}$$

we have ISS for each subsystem with $x_i(t) \le e^{-t}x_j^0 + \gamma_i(||x_i||_\infty) + \eta_i(||u||_\infty)$, where $\gamma_i(s) < s$ and hence $\gamma_1 \circ \gamma_2(s) < s$ for $s > 0$, but there are solutions $x_1 = x_2 = const$ given by

$$\dot{x}_1 = -x_2 e^{-x_2} + u, \quad \text{with } u = x_2 e^{-x_2}.$$

Here $x_1 = x_2$ can be chosen arbitrary large with $u \to 0$ for $x_1 \to \infty$, so that the system cannot satisfy the asymptotic gain property and is therefore not ISS. Hence the condition $\Gamma(s) \not\geq s$, for all $s \in \mathbb{R}^n_+ \setminus \{0\}$, or for two subsystems $\gamma_{12} \circ \gamma_{21}(s) < s$, for all $s > 0$, is not sufficient for the ISS of the composite system in the nonlinear case.

Application to linear systems Linear systems are an important special case for which the results are applicable. Consider the following setup where in the sequel we omit the external input for notational simplicity. Let

$$\dot{x}_j = A_j x_j, \quad x_j \in \mathbb{R}^{N_j}, \quad j = 1, \dots, n, \tag{50}$$

describe n globally asymptotically stable linear systems, which are interconnected through

$$\dot{x}_j = A_j x_j + \sum_{k=1}^{n} \Delta_{jk} x_k, \quad j = 1, \dots, n, \tag{51}$$

which can be rewritten as

$$\dot{x} = (A + \Delta)x, \tag{52}$$

where A is block diagonal, $A = \operatorname{diag}(A_j, j = 1, \dots, n)$, each A_j is Hurwitz and the matrix $\Delta = (\Delta_{jk})$ is also in block form and encodes the connections between the n subsystems. We suppose that $\Delta_{jj} = 0$ for all j. Define the matrix $R = (r_{jk})$, $R \in \mathbb{R}^{n \times n}_+$, by $r_{jk} := ||\Delta_{jk}||$. For each subsystem, there exist positive constants M_j, λ_j, such that $\|e^{A_j t}\| \le M_j e^{-\lambda_j t}$ for all $t \ge 0$.

Define a matrix $D \in \mathbb{R}^{n \times n}_+$ by $D := \operatorname{diag}(\frac{M_j}{\lambda_j}, j = 1, \dots, n)$. It is easy to see that in this case the gain matrix is $\Gamma^{\text{ISS}} = DR$. Then from Corollary 16 we obtain

Corollary 19 *If $\rho(D \cdot R) < 1$ then (52) is globally asymptotically stable.*

Note that this is a special case of a theorem, which can be found in Vidyasagar [32, p. 110], see Remark 17. The corollary is also a consequence of more general and precise results of a recent paper by Karow et al. [16].

5.2 Topological consequences of the small gain condition

For the following statement we define the open domains

$$\Omega_i = \left\{ s \in \mathbb{R}^n_+ \, : \, s_i > \sum_{j \neq i} \gamma_{ij}(s_j) \right\}, \quad i = 1, \ldots, n.$$

Note that $\Omega_i = \{s \in \mathbb{R}^n_+ \, : \, s_i > \Gamma(s)_i\}$. Also we need the simplex Δ_r defined as the intersection of the positive orthant $\mathbb{R}^n_+$ with the hyperplane $s_1 + \cdots + s_n = r > 0$. The vertices of Δ_r are given by $re_1, \ldots, re_n$ (where e_i denotes the i-th unit vector). The convex hull of a set C is denoted by $\operatorname{conv} C$.

Proposition 20 *Consider $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$ with zero diagonal. Then condition (44) is equivalent to $\bigcup_{i=1}^n \Omega_i = \mathbb{R}^n_+ \setminus \{0\}$. Furthermore (44) implies that for all $r > 0$*

$$\Delta_r \cap \bigcap_{i=1}^n \Omega_i \neq \emptyset. \tag{53}$$

Proof Let $s \neq 0$. Formula (44) is equivalent to the existence of at least one index $i \in \{1, \ldots, n\}$ with $s_i > \sum_{j \neq i} \gamma_{ij}(s_j)$. This proves the first part of the assertion.

To prove the second part we will use the Knaster–Kuratovski–Mazurkiewicz (KKM) theorem (see [11]), a topological fixed point theorem for simplices. The KKM theorem states that if for any face of Δ_r given by $\sigma(re_{i_1}, \ldots, re_{i_k}) = \operatorname{conv}\{re_{i_1}, \ldots, re_{i_k}\}$, $k \in \{1, \ldots, n\}$, $1 \leq i_1 < i_2 < \cdots < i_k \leq n$, we have

$$\sigma(re_{i_1}, \ldots, re_{i_k}) \subset \cup_{j=1}^k \Omega_{i_j},$$

then (53) follows (where we use that $\Delta_r \cap \Omega_i$ is open in Δ_r).

Without loss of generality consider the face $\sigma = \operatorname{conv}\{re_1, \ldots, re_k\}$ (the other cases follow by permutation) and let $s \in \sigma$. By assumption $\Gamma(s) \not\geq s$, so there is an index i such that $\Gamma(s)_i < s_i$. As $\Gamma(s) \geq 0$ it follows that for this index $s_i > 0$. Hence $1 \leq i \leq k$. This shows $s \in \Omega_i$ for some $1 \leq i \leq k$ and the assumptions of the KKM theorem are satisfied. This completes the proof.

We need the following invariance property of $\cap_i \Omega_i$.

Lemma 21 *Consider $\Gamma \in (\mathcal{K} \cup \{0\})^{n \times n}$ with zero diagonal and assume (44). Assume that Γ has no zero row. If $s > 0$ satisfies $s \in \cap_i \Omega_i$, then $\Gamma(s) \in \cap_i \Omega_i$.*

Proof First note, that as $s > 0$ and as Γ has no zero rows, it follows that $\Gamma(s) > 0$. Furthermore, $s \in \cap_i \Omega_i$ is equivalent to $\Gamma(s) < s$. Applying Γ to this inequality it follows from monotonicity of Γ that $\Gamma^2(s) < \Gamma(s)$, which is equivalent to $\Gamma(s) \in \cap_i \Omega_i$.

Note in particular that the set $\bigcap_{i=1}^n \Omega_i$ is unbounded. It is of further interest to know that it is unbounded in all components. This will provide the key argument in the stability analysis of the associated discrete time system $s(k+1) = \Gamma(s(k))$. However, this is not true in general. We therefore note

Proposition 22 *Consider $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ with zero diagonal and assume (39). If Γ is irreducible, then for any $s \in \mathbb{R}^n_+$ there is a $z \in \cap_i \Omega_i$ such that $z \geq s$.*

Proof We first assume that Γ is primitive, see the Appendix. Let k_0 be the non-negative integer given by Lemma 26a), such that $P_{\{i\}} \circ \Gamma^{k_0} \circ R_{\{j\}} \in \mathcal{K}_\infty$ for any i, j, where P denotes a projection, R an injection. See Sect. 2 for the definitions of P and R. Fix $s \geq 0$. For $t \in [0, \infty)$, $i = 1, \dots, n$ we define

$$\Gamma_i^{k_0}(t) := \Gamma^{k_0} \circ R_{\{i\}}(t).$$

As $P_{\{i\}} \circ \Gamma^{k_0} \circ R_{\{j\}} \in \mathcal{K}_\infty$ for all $i, j = 1, \dots, n$, there is a $T \in \mathbb{R}_+$ such that for all $i = 1, \dots, n$ we have

$$\Gamma_i^{k_0}(t) > s \quad \text{for all} \quad t \geq T. \tag{54}$$

Choose $r = nT$ and $v > 0$, $v \in \Delta_r \cap (\cap_i \Omega_i)$. Such a v exists as the intersection $\cap_i \Omega_i$ is open in Δ_r. Then $v_i \geq T$ for some $1 \leq i \leq n$ and so

$$\Gamma^{k_0}(v) \geq \max_j \Gamma_j^{k_0}(v_j) \geq \Gamma_i^{k_0}(v_i) \geq \Gamma_i^{k_0}(T) > s.$$

By Lemma 21 we have $\Gamma^{k_0}(v) \in \cap_i \Omega_i$. This completes the proof for the case that Γ is primitive.

In the case that Γ is not primitive we apply Lemma 26b). So without loss of generality, we have a block-diagonal power Γ^ν of Γ, where each of the square blocks on the diagonal is primitive. Then arguing as before, for every s we can choose a $v > 0$, $v \in \cap_i \Omega_i$ so that

$$\Gamma^{k_0 \nu}(v) > s.$$

This completes the proof.

Let us briefly explain a further reason, why the overlapping condition (53) is interesting, apart from the fact, that Proposition 22 is important for the results of the next section. From the theory of ISS-Lyapunov functions (see, e.g., [14]) it is known, that a system of the form (3) is ISS if and only if it has a smooth ISS-Lyapunov function. In the context of n interconnected systems the small gain condition states, according to Proposition 20, that along trajectories of the interconnection

(i) in every state there is one subsystem with a decaying Lyapunov function,
(ii) there is an unbounded region, where the Lyapunov functions of all subsystems decay.

In [6] it is shown how this observation leads to the explicit construction of a Lyapunov function for the interconnection. A preliminary version can be found in [4].

A typical situation in case of three one dimensional systems ($\mathbb{R}^3$) is presented in Fig. 3 on a plane crossing the positive semi axis. The three sectors are the intersections of the sets Ω_i with this plane.

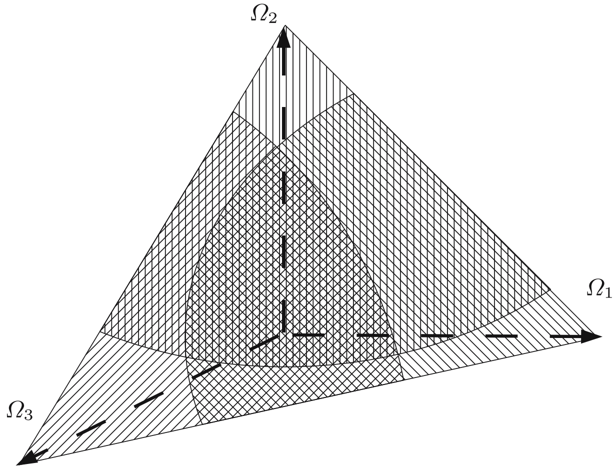

Fig. 3 Overlapping of Ω_i domains in $\mathbb{R}^3$

5.3 Stability of monotone discrete time systems

We now relate the small gain condition to stability properties of the monotone discrete time system $s_{k+1} = \Gamma(s_k)$ defined on the state space $\mathbb{R}^n_+$.

In recent papers by Angeli, De Leenheer, and Sontag small gain type theorems for continuous time monotone systems are presented, where the main stability condition is that an associated discrete time system has a unique globally attractive fixed point, see [1, Theorem 3] or [2, Theorem 1]. These results extend easily to arbitrary interconnections of more than two systems, and the stability conditions remain the same. This extension is implicitly contained in the result [8, Theorem 2].

In this section we need the notion of irreducibility. In Appendix the necessary definitions are recalled and a dichotomy lemma for irreducible matrices is stated.

A related question to the stability of the composite system (17) is, whether or not the discrete time positive dynamical system defined by

$$s_{k+1} = \Gamma(s_k), \quad k = 1, 2, \dots \tag{55}$$

with initial state $s_0 \in \mathbb{R}^n_+$ has $x^* = 0$ as asymptotically stable and globally attractive fixed point.

Theorem 23 *Assume that Γ is irreducible. Then the equilibrium point* 0 *of system* (55) *is globally asymptotically stable if and only if $\Gamma(s) \not\geq s$ for all $s \in \mathbb{R}^n_+ \setminus \{0\}$.*

Proof If $\Gamma(s_0) \geq s_0$ for some $s_0 \in \mathbb{R}^n_+ \setminus \{0\}$ then by monotonicity of Γ we have $\Gamma^k(s_0) \geq \Gamma^{k-1}(s_0) \geq s_0$ for $k = 2, 3, \dots$. Hence the sequence $\{\Gamma^k(s_0)\}_{k=0}^{\infty}$ does not converge to 0 as $k \to \infty$. Hence $x^* = 0$ is not globally attractive.

Conversely, if $\Gamma \not\geq \mathrm{id}$, then by Proposition 22 for every $s \geq 0$ there is a $z \in \cap_i \Omega_i$ with $s < z$. This implies $0 \leq \Gamma^k(s) \leq \Gamma^k(z)$ for all $k \in \mathbb{N}$. So to prove global attractivity of $x^* = 0$ all we have to show is that $\Gamma^k(z) \to 0$ for $z \in \cap_i \Omega_i$. For

$z \in \cap_i \Omega_i$ we have by induction that $\Gamma^{k+1}(z) \leq \Gamma^k(z)$ for all $k \in \mathbb{N}$. This implies $w := \lim_{k\to\infty} \Gamma^k(z)$ exists. By continuity of Γ it follows that $\Gamma(w) = w$. As $\Gamma \not\geq \mathrm{id}$ this implies $w = 0$. Finally, to prove stability of $x^* = 0$, fix $\varepsilon > 0$, and choose $z > 0, z \in \Delta_\varepsilon \cap (\cap_i \Omega_i)$. Then for all $s < z$ we have $\Gamma^k(s) \leq \Gamma^k(z) < z$ for all $k \in \mathbb{N}$. Now choose $\delta > 0$, such that $|s|_{\max} \leq \delta$ implies $s < z$. Then $|s|_{\max} \leq \delta$ implies $|\Gamma^k(s)|_{\max} < |z|_{\max} < \varepsilon$ for all $k \geq 0$. This concludes the proof.

In [5] it is shown, that the condition, that there exists a diagonal operator $D : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ as in (19) such that $(D \circ \Gamma)(s) \not\geq s$ for all $s \neq 0$, $s \in \mathbb{R}^n_+$, is in fact equivalent to global asymptotic stability of the fixed point $x^* = 0$ of $\tilde{D} \circ \Gamma$, for a certain diagonal operator $\tilde{D}$ as in (19).

The following example shows that irreducibility is essential in the statement of Theorem 23.

Example 24 Consider the map $\Gamma : \mathbb{R}^2_+ \to \mathbb{R}^2_+$ defined by

$$\Gamma := \begin{bmatrix} \gamma_{11} & \mathrm{id} \\ 0 & \gamma_{22} \end{bmatrix}$$

where for $t \in \mathbb{R}_+$

$$\gamma_{11}(t) := t(1 - e^{-t})$$

and the function γ_{22} is constructed in the sequel. First note that $\gamma_{11} \in \mathcal{K}_\infty$ and $\gamma_{11}(t) < t, \forall t > 0$. Let $\{\varepsilon_k\}_{k=1}^\infty$ be a strictly decreasing sequence of positive real numbers, such that $\lim_{k\to\infty} \varepsilon_k = 0$ and $\lim_{K\to\infty} \sum_{k=1}^K \varepsilon_k = \infty$. For $k = 1, 2, \ldots$ define

$$\gamma_{22}\left(\varepsilon_k + \left(1 + \sum_{j=1}^{k-1} \varepsilon_j\right) e^{-\left(1+\sum_{j=1}^{k-1} \varepsilon_j\right)}\right) := \varepsilon_{k+1} + \left(1 + \sum_{j=1}^{k} \varepsilon_j\right) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)}$$

and observe that

$$\varepsilon_k + \left(1 + \sum_{j=1}^{k-1} \varepsilon_j\right) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} > \varepsilon_{k+1} + \left(1 + \sum_{j=1}^{k} \varepsilon_j\right) e^{-(1+\sum_{j=1}^{k} \varepsilon_j)},$$

since $\varepsilon_k > \varepsilon_{k+1}$ for all $k = 1, 2, \ldots$ and the map $t \mapsto t \cdot e^{-t}$ is strictly decreasing on $(1, \infty)$.

Moreover we have by assumption, that

$$\varepsilon_k + \left(1 + \sum_{j=1}^{k-1} \varepsilon_j\right) e^{-(1+\sum_{j=1}^{k-1} \varepsilon_j)} \xrightarrow[k\to\infty]{} 0.$$

These facts together imply that γ_{22} may be extrapolated to some $\mathcal{K}_\infty$-function, in a way such that $\gamma_{22}(t) < t$, $\forall t > 0$ holds.

Note that by our particular construction we have $\Gamma(s) \ngeq s$ for all $s \in \mathbb{R}^2_+ \setminus \{0\}$. Now define $s^1 \in \mathbb{R}^2_+$ by

$$s^1 := \begin{bmatrix} 1 \\ 1 + e^{-1} \end{bmatrix}$$

and for $k = 1, 2, \ldots$ recursively define $s^{k+1} := \Gamma(s^k) \in \mathbb{R}^2_+$.

By induction one verifies that

$$s^{k+1} = \Gamma^k(s^1) = \begin{bmatrix} 1 + \sum_{j=1}^k \varepsilon_j \\ \varepsilon_{k+1} + (1 + \sum_{j=1}^k \varepsilon_j) e^{-(1+\sum_{j=1}^k \varepsilon_j)} \end{bmatrix}.$$

By our previous considerations and assumptions we easily obtain that the second component of the sequence $\{s^k\}_{k=1}^\infty$ strictly decreases and converges to zero as k tends to infinity. But at the same time the first component strictly increases above any given bound.

Hence we established that $\Gamma(s) \ngeq s \; \forall s \neq 0$ in general does not imply $\forall s \neq 0: \; \Gamma^k(s) \to 0$ as $k \to \infty$. For this example it is also easy to verify that $\cap_i \Omega_i$ is not unbounded in all components. So that the assertion of Proposition 22 is false in this case.

Remark 25 Note that we can even turn the constructed 2×2 matrix Γ into the null-diagonal form to conform with the structure of gain matrices in Sect. 3. Using the same notation for γ_{ij} as in Example 24, we just define

$$\Gamma := \begin{bmatrix} 0 & \gamma_{11} & \mathrm{id} & 0 \\ \gamma_{11} & 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 & \gamma_{22} \\ 0 & 0 & \gamma_{22} & 0 \end{bmatrix} \quad \text{and} \quad s^1 := \begin{bmatrix} 1 \\ 1 \\ 1 + e^{-1} \\ 1 + e^{-1} \end{bmatrix}$$

and easily verify that $\Gamma^k(s^1)$ does not converge to 0.

5.4 Summary map of the interpretations concerning Γ

In Fig. 4 we summarize the relations between various statements about Γ that were proved in Sect. 5.

6 Conclusions

We have considered a composite system consisting of an arbitrary number of nonlinear arbitrarily interconnected input-to-state stable subsystems, as they arise in applications.

For this general case a network version of the *nonlinear small gain theorem* has been obtained. For linear interconnection gains this is a special case of a known result,

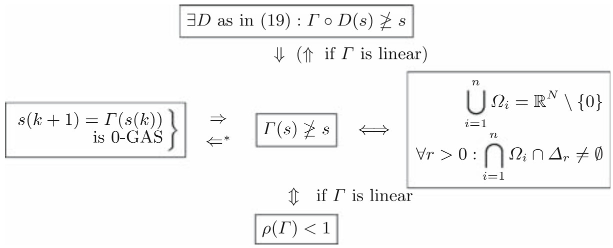

Fig. 4 Some implications and equivalences of the generalized small gain condition. All statements are supposed to hold for all $s \in \mathbb{R}^n_+$, $s \neq 0$. The implication denoted by * holds if Γ is linear or irreducible

cf. [32, p. 110]. It has been shown how the generalized small gain theorem can be applied to the analysis of linear systems and further implications of the small gain condition have been discussed to clarify its significance. The problem of constructing Lyapunov functions within this framework will be dealt with in [6].

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Appendix: Non-negative matrices and graphs

A (finite) directed graph $G = \{V, E\}$ consists of a set of vertices V and a set of edges $E \subset V \times V$. We may identify $V = \{1, \dots, n\}$ in case of n vertices. The adjacency matrix $A_G = (a_{ij})$ of this graph is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (j, i) \in E, \\ 0 & \text{else.} \end{cases}$$

Conversely, given an $n \times n$-matrix A, a graph $G(A) = \{V, E\}$ is defined by $V := \{1, \dots, n\}$ and $E = \{(j, i) \in V \times V : a_{ij} \neq 0\}$.

There are several concepts and results of (non-negative) matrix theory, which are of purely graph theoretical nature. Hence the same can be done for our gain matrix Γ. We may associate a graph $G(\Gamma)$, which represents the interconnections between the subsystems, in the same manner, as we would do for matrices.

The graph of a power Γ^k of Γ consists of the same set of vertices V as Γ and has edges $E = \{(j, i) \in V \times V :$ Component j influences component $i\}$. This can be stated equivalently as $E = \{(j, i) \in V \times V : \exists s \in \mathbb{R}^n_+ : t \mapsto P_{\{i\}} \circ \Gamma^k(s + t \cdot e_j)$ is of class $\mathcal{K}\}$ or $E = \{(j, i) \in V \times V : \forall s \in \mathbb{R}^n_+ : t \mapsto P_{\{i\}} \circ \Gamma^k(s + t \cdot e_j)$ is of class $\mathcal{K}\}$. With this notation we have

$$A(\Gamma^k) = A(G((A(\Gamma))^k)), \tag{56}$$

where the right hand side denotes the adjacency matrix of the graph associated to the matrix $(A(\Gamma))^k$, that is the graph with edges $E = \{(i, j) \in V \times V : (A(\Gamma))^k)_{ij} \neq 0\}$.

We say Γ is *irreducible*, if $G(\Gamma)$ is *strongly connected*, that is, for every pair of vertices (i, j) there exists a sequence of edges (a *path*) connecting vertex i to vertex j. Obviously Γ is irreducible if and only if Γ^T is. Γ is called *reducible* if it is not irreducible.

The gain matrix Γ is *primitive*, if there exists a positive integer m such that $(A_{G_\Gamma})^m$ has only positive entries.

For the following facts only the graph structure associated to a gain matrix is relevant.

If Γ is reducible, then a permutation transforms it into a block upper triangular matrix. From an interconnection point of view, this splits the system into cascades of subsystems each with irreducible (or zero) adjacency matrix.

Lemma 26 *Assume the gain matrix Γ is irreducible. Then there are two distinct cases:*

(a) *The gain matrix $\Gamma = (\gamma_{ij}(\cdot))$, where $\gamma_{ij}(\cdot) \in \mathcal{K}$ or $\gamma_{ij} = 0$, is primitive and hence there is a non-negative integer k_0 such that Γ^{k_0} fulfills $P_{\{i\}} \circ \Gamma^{k_0} \circ R_{\{j\}} \in \mathcal{K}$ for any i, j.*

(b) *The gain matrix Γ can be transformed to*

$$P \Gamma P^T = \begin{pmatrix} 0 & \tilde{\Gamma}_{12} & 0 & \dots & 0 \\ 0 & 0 & \tilde{\Gamma}_{23} & \dots & 0 \\ \vdots & & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \tilde{\Gamma}_{\nu-1,\nu} \\ \tilde{\Gamma}_{\nu 1} & 0 & 0 & \dots & 0 \end{pmatrix} =: \tilde{\Gamma} \tag{57}$$

using some permutation matrix P, where the zero blocks on the diagonal are square and where the adjacency matrix of $\tilde{\Gamma}^\nu$ is of block diagonal form with square primitive blocks on the diagonal. Here ν is the index of imprimitivity, which is the number of nonzero blocks in the above definition of $\tilde{\Gamma}$.

Proof Let A_{G_Γ} be the adjacency matrix corresponding to the graph associated with Γ. This matrix is primitive if and only if Γ is primitive. Note that the (i, j)th entry of $A^k_{G_\Gamma}$ is zero if and only if the (i, j)th entry of Γ^k is zero. Multiplication of Γ by a permutation matrix only rearranges the positions of the class $\mathcal{K}$-functions, hence this operation is well defined. From these considerations it is clear, that it is sufficient to prove the lemma for the matrix $A := A_{G_\Gamma}$. But for non-negative matrices this result follows from standard results in the theory of non-negative matrices, see, e.g., [3] or [18].

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