State Dependent AIMD Algorithms and Consensus Problems

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This papers analyses a class of nonlinear positive systems that model the dynamics of nonlinear *additive-increase multiplicative-decrease* (AIMD) protocols. The system class covers a range of protocols that are currently used in real communication networks, such as standard TCP, and recent proposals for congestion control protocols such as Scalable TCP.

1 Introduction: Linear AIMD Congestion Control

The standard linear AIMD algorithm employed in TCP (transmission control protocol) describes probing strategies that evolve in cycles, each cycle having two phases. The first phase of the cycles is instantaneous. It occurs when capacity is reached, users are notified and each responds by down-scaling its utilization-rate (abruptly) by a multiplicative factor. This phase is called the *Multiplicative Decrease* (MD) phase. During the second phase of a cycle, each user increases his utilization-rate linearly until congestion is reached again, at which point the first phase of the next cycle is entered. The second phase is called the *Additive Increase* (AI) phase.

Denote the share of the collective resource allocated to user i at time t by $x_i(t)$ and let $x(t) = [x_1(t), \dots, x_n(t)]^T$. The capacity constraint requires that $\sum_{i=1}^n x_i(t) < C$, with C as the total capacity of the resource available to the entire system.

The k^{th} cycle begins at a time $t(k)$ at which the global utilization of the resource reaches capacity. The instantaneous decrease of the utilization-rate of user i during the MD phase of the k^{th} cycle is expressed by:

$$x_i(t(k)^+) = \beta_i x_i(t(k)^-), \tag{1}$$

where $x_i(t(k)^+) := \lim_{t \searrow t(k)} x_i(t)$, $x_i(t(k)^-) := \lim_{t \nearrow t(k)} x_i(t)$ and β_i is a constant in the open interval $(0, 1)$. During the AI phase of the k^{th} cycle, the utilization-rate of user i evolves according to:

$$x_i(t) = x_i(t(k)^+) + \alpha_i(t - t(k)), \tag{2}$$

where α_i is a positive constant. The $(k+1)^{st}$ cycle begins at time $t(k+1)^+$ that equals the time t for which the right hand-side of (2) reaches capacity. Combining (1) and (2), we see that the evolution of the utilization-rate of user i between the k^{th} and $(k+1)^{st}$ congestion points is given by

$$x_i(t(k+1)) = \beta_i x_i(t(k)) + \alpha_i \Big(t(k+1) - t(k) \Big), \tag{3}$$

This avenue of investigation is explored e.g. in [2] where it is shown that the transformation of the utilization-rates between consecutive congestion points is linear with the representation $x(k+1) = Ax(k)$, where the matrix A is derived from (3).

2 Nonlinear AIMD Congestion Control

We next describe a nonlinear variant of the basic AIMD algorithm which coincides with the standard (linear) AIMD except that in the AI phase the increase in the utilization-rate of each user i is dictated by the nonlinear function $x_i(k) \mapsto x_i(k)^{\kappa_i}$, where $\kappa_i \in \mathbf{R}$. This form is motivated by the desire to make TCP more aggressive in high-speed and long distance networks. The evolution of the utilization-rate of user i between the k^{th} and $(k+1)^{st}$ congestion points is then given by

$$x_i(k+1) = \beta_i x_i(k) + (x_i(k))^{\kappa_i}(t(k+1) - t(k)), \tag{4}$$

which replaces (3). This family of congestion control protocols includes standard TCP when $\kappa = 0$, Scalable TCP when $\kappa = 1$, and several other proposed algorithms for high speed networks.

It has been recently shown by several authors that some choices of the nonlinearities lead to poor dynamic properties, including the lack of stable utilization-rates; see [1]. It is therefore of interest to determine the properties of networks with various values of κ.

For reasons of space we are only able to state results. We refer to [3] for complete proofs.

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3 Stability Results: Fixed Points

We consider the system (4) subject to the constraint that $\sum_{i=1}^n x_i(k) = C$ for all k, where $\beta_i \in (0,1)$ and $C > 0$ denotes the capacity of the resource. The index k enumerates the time instances at which the capacity constraint is reached and $\beta_i \in (0,1)$ is the backoff parameter applied when the i'th source is informed of 'congestion'. The time T between congestion event depends on the state at the beginning of the event and is given by $(t(k+1) - t(k)) = T(x(k))$, where

$$T(x) = \frac{C - \sum_{j=1}^n \beta_j x_j}{\sum_{j=1}^n x_j^{\kappa_j}} = \frac{\sum_{j=1}^n (1-\beta_j) x_j}{\sum_{j=1}^n x_j^{\kappa_j}} . \tag{5}$$

Definition 3.1 (Stable Networks) A network given by dynamics of the form (4) on $\tilde{\Sigma}$ is called stable if (i) system (4) has a unique equilibrium point $w^* \in \tilde{\Sigma}$, and (ii) w^* is a globally asymptotically stable fixed point of (4) on $\tilde{\Sigma}$. A network is said to be unstable if it is not stable.

Let us first determine the fixed points of (4).

Proposition 3.2 *Consider the system* (4), (5). *Then*

(i) if $(1-\kappa_i)(1-\kappa_j) > 0$, for all $i, j \in \{1, \dots, n\}$ then there exists a unique fixed point $w^* \in \tilde{\Sigma}$. This fixed point is determined by the equations for $i = 1, \dots, n$

$$w_i^* = \left(\frac{T(w^*)}{(1-\beta_i)} \right)^{1/(1-\kappa_i)} , \quad \text{where } T(w^*) > 0 \text{ uniquely solves} \quad C = \sum_{j=1}^n \left(\frac{T^*}{(1-\beta_j)} \right)^{1/(1-\kappa_j)} . \tag{6}$$

(ii) if $\kappa_i \neq 1, i = 1, \dots, n$ and there are $i, j \in \{1, \dots, n\}$, such that $(1-\kappa_i)(1-\kappa_j) < 0$ then the system is generically unstable, that is, there exists a constant $C^* > 0$, such that the system (4), (5) *has no fixed point in $\tilde{\Sigma}$, if $0 < C < C^*$ and multiple fixed points, if $C > C^*$.*

(iii) if exactly one source i satisfies $\kappa_i = 1$, then there exists a unique fixed point $w^* \in \tilde{\Sigma}$ if and only if

$$C - \sum_{j \neq i} \left(\frac{1-\beta_i}{1-\beta_j} \right)^{1/(1-\kappa_j)} > 0 . \tag{7}$$

(iv) if two or more sources $i_1, \dots, i_l$ have the parameter $\kappa_{i_1} = \ldots = \kappa_{i_l} = 1$, then the network is unstable.

4 TCP and consensus

We have so far obtained conditions for the existence of a unique fixed point of the network in terms of the κ_i, namely all κ_i have to be chosen such that the sign of $1 - \kappa_i$ is fixed, (ignoring a few special cases, that are of no particular relevance to the general design of protocols). In [3] it is shown that further restrictions have to be placed on the κ_i in order to obtain local asymptotic stability of the unique fixed points. In particular, for local stability it is necessary that $\kappa_i < 1$ for $i = 1, \dots, n$. To speak of a stable network, however, we would like to achieve global stability. This is the topic of the following result.

Proposition 4.1 *Let $\kappa \in [0,1)$, $\kappa_i = \kappa$, $\beta_i \in [0,1), i = 1, \dots, n$ and $C > 0$. Then the fixed point $w^* = (w_1^*, \dots, w_n^*)$ given by* (6) *is globally asymptotically stable for system (4).*

The proof of this result is closely related to stability results that have been recently obtained in the framework of consensus problems. In this sense the stability property of AIMD algorithms may be interpreted as the problem of the sources reaching consensus on the share of the resource that each obtains without direct communication between the different users.

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